Solutions to Selected Even Numbered Exercises in Chapter 3

Section 3.1.

\# 2. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are examples of linearly independent vectors which are not orthogonal.

Conversely, we note that a set of mutually orthogonal vectors are linearly independent. To see this, let $v_1, v_2, \ldots, v_n$ be mutually orthogonal, -i.e., $v_i^T v_j = 0$ if $i \neq j$. Consider $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$. When we take the inner product on both sides with v_1, we see that on the right

$$v_1^T 0 = 0$$

and on the left

$$v_1^T (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 v_1^T v_1 + c_2 v_1^T v_2 + \cdots c_n v_1^T v_n = c_1 v_1^T v_1.$$

This shows that

$$c_1 v_1^T v_1 = 0$$

and since $v_1^T v_1 \neq 0$, we must have $c_1 = 0$. Repeat the same argument with v_1 replaced by $v_2, \ldots, v_n$, we get $c_2 = \cdots = c_n = 0$. This show that $v_1, v_2, \ldots, v_n$ are linearly independent.

\# 6. Let (x, y, z) be a vector orthogonal to $(1, 1, 1)$ and $(1, -1, 0)$. This means that after taking the inner products

$$\begin{cases} x + y + z & = 0 \\ x - y & = 0 \end{cases}$$

Solving this system

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 1 & -1 & 0 & | & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & 0 \\ 0 & 1 & \frac{1}{2} & | & 0 \end{bmatrix}$$

we get $x = -\frac{1}{2}c, y = -\frac{1}{2}c, z = c$. In particular with $c = 2$, $x = -1, y = -1, z = 2$ or $(-1, -1, 2)$.

Now, since $(1, 1, 1)$, $(1, -1, 0)$ and $(-1, -1, 2)$ are orthogonal, we can obtain an orthonormal basis by normalizing each vector. Namely

$$\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0), \frac{1}{\sqrt{6}}(-1, -1, 2)\}$$

Section 3.2

12. The direction of the line $x + 2y = 0$ can be represented by a vector $a = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Using the formula for the projection $\frac{aa^T}{a^Ta}$, we get

$$\frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

22. From #21, we see that

$$P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \quad \text{and} \quad P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

Hence

$$(P_1 + P_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix}.$$

Now since $a_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $P_3 = \frac{a_3 a_3^T}{a_3^T a_3}$ yields

$$P_3 = \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}.$$

Also

$$P_1 + P_2 + P_3 = I_3.$$

Section 3.4

\# 2. Here one may compute the projections of b onto a_1 and a_2 by constructing the matrices of projections and in the previous exercises, or simply use the formula for the projected vector, we get

$$P_{a_1}b = (b^T a_1)a_1 = 2\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \qquad P_{a_2}b = (b^T a_2)a_2 = 2\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

Note that the projection of b onto a is given by $\frac{(b^T a)a}{a^T a}$ but if $\|a\| = 1$, then it reduces to $(b^T a)a$. Similarly

$$P_{a_3}b = (b^T a_3)a_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

and $(P_1 + P_2 + P_3)b = b$ or $(P_1 + P_2 + P_3) = I$. This is because a_1, a_2, a_3 are linearly independent vectors, each having dimension 3 in R^3, hence they form a basis for R^3 and thus span R^3. Hence any vector b in R^3, when projected onto R^3, must be itself.

\# 14. This was done in class.

\# 16. Following the technique of Gram Schmidt orthogonalization process, we get

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$