Solutions to Selected Even Numbered Exercises in Sections 5.1 and 5.2

Section 5.1.

\# 2.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \Longrightarrow det[A - \lambda I] = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$$

Hence, 3 and 2 are the eigenvalues of A. When $\lambda = 3$, solving

$$\begin{bmatrix} 1-\lambda & -1 & | & 0 \\ 2 & 4-\lambda & | & 0 \end{bmatrix}$$

, we get an eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Similarly, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$. Also, from

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

we get $c_1 = -6$ and $c_2 = 6$. Hence

$$u(t) = (-6)e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

\# 4. Here, eigenvalues of $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ are $\lambda = 0, 1$ and its corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. Also, solving

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

we get $c_1 = 4$ and $c_2 = 1$. Hence

$$u(t) = 4e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

\# 12. Eigenvalues of $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ are $\lambda = 5, -5$ and its corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively.

$$det \begin{bmatrix} a-\lambda & b \\ b & a-\lambda \end{bmatrix} = \lambda^2 - 2a\lambda + (a^2 - b^2)$$

and the roots of this quadratic polynomial are $a \pm b$. With $\lambda = a + b$ a corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, for $\lambda = a - b$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

\# 20. For $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, eigenvalues are $\lambda = 5, -1$ and its corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ respectively. Since $B = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} = A + I$, eigenvalues of B are different from those of A by 1, namely $\lambda = 6, 0$ with the same eigenvectors as A.

\#22. As we see in #24, if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2 with same eigenvectors. Eigenvalues of $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ are 2 and -3 with corresponding eigenvectors of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ respectively. Also $A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$ have eigenvalues 4 and 9 with the same eigenvectors.

\# 24.

(a) Since $Ax = \lambda x$, applying A to both sides

$$A^2 x = A(Ax) = A(\lambda x) = \lambda A x = \lambda^2 x$$

(b) Since $Ax = \lambda x$, applying A^{-1} to both sides,

$$A^{-1}Ax = A^{-1}(\lambda x) \Longrightarrow x = \lambda A^{-1} x \Longrightarrow A^{-1}x = (\lambda)^{-1} x.$$

Note that if A^{-1} exists, -i.e., A is invertible, then all eigenvalues of A are different from 0, so $(\lambda)^{-1} = \frac{1}{\lambda}$ is valid.

(c) From $Ax = \lambda x$ and $Ix = x$, add Ix to Ax and x to λx, we get $(A+I)x = (\lambda+1)x$.

\# 26. $Q - \lambda I == \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - 2\cos\theta\lambda + 1$. Using the quadratic formula, we find the roots of the last quadratic equation to be

$$\frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i \sin\theta$$

Eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ for $\lambda = \cos\theta + i\sin\theta$ and $\begin{bmatrix} i \\ -1 \end{bmatrix}$ for $\lambda = \cos\theta - i\sin\theta$.

Section 5.2

2. A can be found from $A = S\Lambda S^{-1}$ where $S = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

$$A = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}.$$

6.

(a) As $A^2 = I$, $det A^2 = det I = 1$. But $det A^2 = (det A)^2$ so $(det A)^2 = 1$ which implies $det A = \pm 1$.

(b) As $A^2 = I$ and $A \neq I$ and $A \neq -I$, $Trace A = \lambda_1 + \lambda_2 = 1 + (-1) = 0$ and $det A = \lambda_1 \lambda_2 = -1$.

(c) Using $A^2 = I$ and $A = \begin{bmatrix} 3 & -1 \\ a & b \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 3 & -1 \\ a & b \end{bmatrix} \begin{bmatrix} 3 & -1 \\ a & b \end{bmatrix} = \begin{bmatrix} 9 - a & -3 - b \\ 3a + ab & -a + b^2 \end{bmatrix}$$

Since the last matrix must be I, $9 - a = 1$ which gives $a = 8$ and $-3 - b = 0$ which gives $b = -3$. Note that with these values of a and b, components on the second row are $3a + ab = 0$ and $-a + b^2 = 1$. Hence the second row of A should be $[\,8 \;\; -3\,]$.

16.

(a) $A^3 = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^3 S^{-1}$

(b) $A^{-1} = (S\Lambda S^{-1})^{-1} = (S^{-1})^{-1}\Lambda^{-1}S^{-1} = S\Lambda S^{-1}$.