Determinants

Permutations: Let $S_n = \{1, 2, \ldots, n\}$. Then a one-to-one and onto map of S_n into S_n is called a **permutation**.

For example, $\sigma_1(1) = 1, \sigma_1(2) = 2$ and $\sigma_2(1) = 2, \sigma_1(2) = 1$ are two permutations of $S_2 = \{1, 2\}$ into itself. Symbolically, we write this as

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

For $S_3 = \{1, 2, 3\}$, there are six permutations. They are

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \qquad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

In general, there are $n!$ permutations of S_n. In the examples above, there are $2! = 2$ permutations for S_2 and $3! = 6$ permutations for S_3.

Inversions in each permutation: In each permutation σ, if $i < j$ and $\sigma(i) > \sigma(j)$, then it is said to be an inversion. For example, in σ_6 above, $\sigma_6(1) = 3 > 2 = \sigma_6(2)$ is an inversion. Similarly, $\sigma_6(1) = 3 > 1 = \sigma_6(3)$ and $\sigma_6(2) = 2 > 1 = \sigma_6(3)$ are also inversions. Hence σ_6 contains three inversions.

Exercises: Specify the number of inversions in $\sigma_1, \ldots, \sigma_5$.

Signs of permutations: Define

$$sign(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has even number of permutations} \\ -1 & \text{if } \sigma \text{ has odd number of permutations} \end{cases}$$

Definition of Determinant: Let A be an $n \times n$ matrix. Then

$$\det A = \sum_{\text{all permutations } \sigma \text{ of } S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

This is the formal definition of determinant of an $n \times n$ matrix A. We see in class that this definition can be used to derive the determinant formulas for a 2×2 matrix

$$det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

as well as the formula for 3×3 matrix given in (2), page 211,

$$det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{array}$$

The list of important properties of the determinant is given in Section 4.2. These properties can be all proved naturally using the formal definition of the determinant given above. In the textbook, these properties are demonstrated in terms of 2×2 matrices.

(1) $detI_n = 1$.
(2) The determinant of a triangular matrix is the product of its diagonal elements.
(3) $detA = 0$ if and only if A is singular.
(4) $det(AB) = det(A)det(B)$.
(5) $det(A^{-1}) = \frac{1}{det(A)}$.
(6) $det(A^T) = det(A)$.

A determinant can be computed by using the definition above, but it is not the most efficient way to compute the determinant. We discuss below two different ways to conveniently compute the determinant; one is the use of row operations and the second the cofactor expansion.

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Row Operations and Determinants: As indicated in the property (2) above, the determinant of a triangular matrix is the product of diagonal elements. Hence, an approach here is to reduce a matrix A to an upper triangular matrix by successive applications of elementary row operations while keeping the records of how each operation is affecting the determinant.

(7) B is obtained from A by interchanging two rows ($R_i \longleftrightarrow R_j$), $det(B) = -det(A)$.
(8) B is obtained from A by multiplying a row of A by λ, ($\lambda R_i \longrightarrow R_i$), $det(B) = \lambda det(A)$.
(9) B is obtained from A by adding a λ multiple of R_i to R_j, ($\lambda R_i + R_j \longrightarrow R_j$), $det(B) = det(A)$.

Example 1:

$$det \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix} \overset{R_1 \leftrightarrow R_4}{=} -det \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R_1 \leftrightarrow R_4}{=} det \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= (4)(1)(2)(2) = 16.$$

Example 2:

$$det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ = \\ R_1 + R_3 \rightarrow R_3 \end{matrix} det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$\begin{matrix} 2R_2 + R_4 \rightarrow R_4 \\ = \end{matrix} det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \begin{matrix} \frac{5}{2}R_3 + R_4 \rightarrow R_4 \\ = \end{matrix} det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 20$$

Exercises: Compute the determinants using the row operations.

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$

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Determinant as Cofactor Expansion: Given an $n \times n$ matrix A, let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and the jth column of A. Then

$$C_{ij} = (-1)^{i+j} det M_{ij}, \qquad i, j = 1, 2, \ldots, n,$$

is called the $i - j$ **cofactor** of A and it is the case that

$$det(A) = \begin{cases} \sum_{j=1}^{n} a_{ij} C_{ij} & \text{for any } i = 1, \ldots, n \text{ - row expansion} \\ \sum_{i=1}^{n} a_{ij} C_{ij} & \text{for any } j = 1, \ldots, n \text{ - column expansion} \end{cases}$$

In the 3×3 matrix above,

$$det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{matrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Exercises: Find the determinants using the cofactor expansion.

1. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$

   (i) Along the first row.
   (ii) Along the second column.

2. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

   (i) Along the first row.
   (ii) Along the first column.

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Applications:

Cramer's Rule: Consider

$$Ax = b,$$

and $A_{b,i}$ denote the $n \times n$ matrix obtained from A by replacing the ith column of A by the vector b, -i.e.,

$$A = [A_1, A_2, \dots, A_i, \dots, A_n] \qquad \text{and} \qquad A_{b,i} = [A_1, A_2, \dots, b, \dots, A_n].$$

Then

$$x_i = \frac{det(A_{b,i})}{det(A)}, \qquad i = 1, 2, \dots, n.$$

Proof:

$$\begin{aligned} det(A_{b,i}) &= det[A_1, A_2, \dots, b, \dots, A_n] \\ &= det[A_1, A_2, \dots, Ax, \dots, A_n] \\ &= det[A_1, A_2, \dots, x_1A_1 + x_2A_2 + \cdots + x_nA_n, \dots, A_n] \\ &= det[A_1, A_2, \dots, x_iA_i, \dots, A_n] \\ &= x_i det[A_1, A_2, \dots, A_i, \dots, A_n] \\ &= x_i det(A). \end{aligned}$$

Exercises: Solve by Cramer's rule.

(i) $\begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$

A Classical Adjoint: Let C denote the matrix of cofactors of A, -i.e., for a 3×3 matrix,

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}.$$

Then the classical adjoint of A, $adj(A)$ is the transpose of C, $adj(A) = C^T$. Also,

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Proof: Consider

$$A^{-1} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nj} & \cdots & m_{nn} \end{bmatrix}$$

Since $AA^{-1} = I_n$,

$$A \begin{bmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{bmatrix} = E_j$$

where E_j is the vector containing 1 in the jth position and zeros elsewhere. By the Cramer's rule,

$$m_{ij} = \frac{det(A_{E_j,i})}{det(A)}.$$

Now,

$$A_{E_j,i} = \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix}$$

Note that

$$det(A_{E_j,i}) = C_{ji},$$

hence

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Exercises: Find the classical adjoint and compute $Aadj(A)$.

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 6 & 6 \end{bmatrix}$