

MATH 316
Introductory Linear Algebra
Fall 2001

SAMPLE FINAL EXAM – SUGGESTED SOLUTIONS

NOTE: These solutions are NOT complete, in most cases they only give the final result. To receive full credit you MUST justify your answers and show ALL your work!

1. We need to find all vectors $\vec{x}$ such that

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \cdot \vec{x} = 0, \quad \begin{pmatrix} 0 \\ 4 \\ 4 \\ -1 \end{pmatrix} \cdot \vec{x} = 0 \text{ and } \begin{pmatrix} 2 \\ 0 \\ -6 \\ 1 \end{pmatrix} \cdot \vec{x} = 0.$$

This is the same as solving the linear system

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & 4 & -1 \\ 2 & 0 & -6 & 1 \end{pmatrix} \vec{x} = \vec{0}.$$

The rref of this matrix is

$$\begin{pmatrix} 1 & 0 & -3 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so the solution (and thus all vectors orthogonal to the three vectors in the problem) can be written as

$$\text{span} \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. (a) Homogeneous systems are always consistent since $\vec{0}$ is always a solution.

- (b) If there are fewer equations than unknowns (i.e., $m < n$), then $\text{rank}(A) < n$, and so the system is either inconsistent or has infinitely many solutions. Using part a) we conclude that the latter must be true.
3. (a) $\det(A) = 1 + b^2 \neq 0$, so A is always invertible.
- (b) $A^{-1} = 1/(1 + b^2) \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}$.
- (c) For $b = 1$ we have that

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix},$$

and so T is a counterclockwise rotation by 45° , followed by a dilation by $\sqrt{2}$.

4.

$$T \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

5. A simple Gauss-Jordan calculation yields that

$$A^{-1} = \begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}$$

6. (a) i. $\vec{0} \in W$ since $0 + 2 \cdot 0 = 0$ and $0 = 0$.
- ii. Let $\vec{u} \in W$ and $\vec{v} \in W$. To see whether $\vec{u} + \vec{v} \in W$ we check $u_1 + v_1 + 2(u_2 + v_2) = (u_1 + 2u_2) + (v_1 + 2v_2) = 0 + 0 = 0$. Also, $u_3 + v_3 = 0 + 0 = 0$, so we conclude that $\vec{u} + \vec{v} \in W$.
- iii. Let $\vec{u} \in W$ and $\lambda \in \mathbb{R}$. Then, for $\lambda\vec{u}$ we obtain $\lambda u_1 + 2\lambda u_2 = \lambda(u_1 + 2u_2) = 0$, and $\lambda u_3 = \lambda \cdot 0 = 0$, so $\lambda\vec{u} \in W$.

We conclude from these three properties that W is a subspace of $\mathbb{R}^3$.

(b) $W = \ker \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$. So $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for W .

(c) Since the basis only has one vector we conclude that $\dim(W) = 1$.

7. We need to compute the rref of A :

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this we see that $\mathcal{B}_{\ker(A)} = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for

$\ker(A)$ and that $\mathcal{B}_{\text{im}(A)} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ is a basis for $\text{im}(A)$.

8. (a) The invertible 3×3 matrices do not form a subspace since the zero matrix is not invertible.

(b) The upper triangular 3×3 matrices form a subspace since the zero matrix is upper triangular, the sum of two upper triangular matrices is upper triangular and the product of a scalar and an upper triangular matrix is upper triangular.

(c) i. Let $\mathbf{0} \in \mathbb{R}^{3 \times 3}$ be the zero matrix. Then $\mathbf{0} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \vec{0}$, so

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \in \ker(\mathbf{0}).$$

ii. Let A and B be in the set. Then

$$(A + B) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = A \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + B \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \vec{0},$$

$$\text{and so } \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \in \ker(A + B).$$

iii. Let A be in the set and $\lambda \in \mathbb{R}$. Then

$$\lambda A \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \lambda \vec{0} = \vec{0},$$

and so λA is also in the set.

From these three properties we conclude that the set of matrices

with $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ in their kernel forms a subspace.

9. (a) $\det(B_T) = 15 \neq 0$, so B_T is invertible and thus T is an isomorphism.
(b)

$$[1+2t+3t^2]_{\mathcal{B}} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix}, \text{ and so } [T(1+2t+3t^2)]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{27}{2} \\ 9 \end{pmatrix}$$

which implies that $T(1+2t+3t^2) = \frac{5}{2}(1-t) + \frac{27}{2}(1+t) + 9t^2 = 16 + 11t + 9t^2$.

10. Consider the matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{pmatrix}$.

- (a) $\det(B) = (bt^2 - tb^2) - (at^2 - ta^2) + (ab^2 - ba^2) = (b-a)t^2 - (b^2 - a^2)t + ab(b-a) = (b-a)[t^2 - (b+a)t + ab] = (b-a)(t-a)(t-b)$.
(b) B is invertible $\det(A) \neq 0$, which happens if $t \neq b$, $t \neq a$ and $a \neq b$.

11. (a) $f_A(\lambda) = (1-\lambda)^3 + 1 + 1 - (1-\lambda) - (1-\lambda) - (1-\lambda) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$.
(b) $f_A(\lambda) = 0$ implies $\lambda_1 = 0$ (alg. multiplicity 2) or $\lambda = 3$ (alg. multiplicity 1).

- (c) $E_0 = \ker(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, so $\lambda = 0$ has geometric multiplicity 2.

$$E_3 = \ker(A - 3I_3) = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \text{ so}$$

$\lambda = 3$ has geometric multiplicity 1.

- (d) Since the geometric multiplicities add up to 3, A is diagonalizable, and

$$S = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

is an example for S (notice the order of the columns in S !).

$$\begin{aligned} 12. \det(A) &= -5 \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} = -5 \left[-1 \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - 3 \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \right] = \\ &= -5(1 - 2) - 15(1 - 2) = -20. \\ \det(A^{-1}) &= 1/\det(A) = -1/20, \det(2A) = 2^4 \det(A) = -320, \text{rank}(A) = \\ &= 4, \text{nullity}(A) = 0. \end{aligned}$$

13. Here is a table with a summary of the results. This is not a full answer, justifications are needed for each answer.

set	rref	lin. ind.?	spans $\mathbb{R}^3$?	basis?
S_1	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	no	yes	no
S_2	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	no	no	no
S_3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	yes	yes	yes
S_4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$	yes	no	no
S_5	$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$	no	no	no
S_6	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	no	no	no

Each one of these sets spans a subspace of $\mathbb{R}^3$.

14. (a) $B\vec{v} = (A^2 - 3A + 2I_n)\vec{v} = A^2\vec{v} - 3A\vec{v} + 2\vec{v} = 4\vec{v} - 6\vec{v} + 2\vec{v} = 0\vec{v} = \vec{0}$, so $\vec{v}$ is an eigenvector of B with associated eigenvalue 0.
- (b) Since B has a 0 eigenvalue it is not invertible. (Or, equivalently, the kernel of B includes $\vec{v} \neq \vec{0}$, so B cannot be invertible.)
15. If $\alpha = 0$, then A is already diagonal. So in parts (a) and (b), we assume that $\alpha \neq 0$.

$$f_A(\lambda) = (1 - \lambda)^2 - \alpha^2. \quad f_A(\lambda) = 0 \implies (1 - \lambda)^2 = \alpha^2 \implies \lambda = 1 \pm \alpha.$$

$$\text{(b)} \quad E_{1+\alpha} = \ker \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

$$E_{1-\alpha} = \ker \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

- (c) A is always diagonalizable since the geometric multiplicities of the eigenvalues always add up to n ($=2$).