

Solutions to Selected Problems from Chapter 1

1.1

10-a

$$\begin{aligned}
 |xy - ab| &= |a(y - b) + b(x - a) + xy - ay - bx + ab| \\
 &\leq |a(y - b)| + |b(x - a)| + |(x - a)(y - b)| \\
 &\leq |a|\epsilon + |b|\epsilon + \epsilon^2 \\
 &= (|a| + |b|)\epsilon + \epsilon^2.
 \end{aligned}$$

1.2

4-b As $x_1 \geq 2$, $x_1 - 1 \geq 1$ which implies $\sqrt{x_1 - 1} \geq 1$ (Exercise #7, 1.3). Hence $x_2 = 1 + \sqrt{x_1 - 1} \geq 2$. Inductively, we see that $2 \leq x_n$, for all $n \in N$. To show $x_{n+1} \leq x_n$ for all $n \in N$, note that, since $x_n - 1 \geq 1$, $\sqrt{x_n - 1} \geq 1$ by Exercise #7, 1.3. By example 1.3, this implies that

$$(1) \quad x_n - 1 \geq \sqrt{x_n - 1}.$$

Now

$$\begin{aligned}
 x_n - x_{n+1} &= x_n - (1 + \sqrt{x_n - 1}) \\
 &= x_n - 1 - \sqrt{x_n - 1} \geq 0 \quad \text{by (1)}.
 \end{aligned}$$

This shows $x_{n+1} \leq x_n$ for all $n \in N$. In particular, $x_2 \leq x_1$ and the problem is solved.

1.3

7-a If $x \neq \sup E$, then there exists $s < x$ such that $a \leq s$ for all $a \in E$. As $x \in E$ by assumption, this implies that $x \leq s$, which contradicts $s < x$ above. Hence $x = \sup E$.

1.4

6-ac (a) $\implies$ (c) Let $E \subseteq X$ and $x \in f^{-1}(f(E))$. Then there exists $y \in f(E)$ such that $f(x) = y$. Now $y \in f(E)$ implies that there exists $x_1 \in E$ such that $f(x_1) = y$. since f is 1-1, $x_1 = x$ which makes $x \in E$. Thus

$$(2) \quad f^{-1}(f(E)) \subseteq E$$

Now, if $x \in E$, $f(x) \in f(E)$ and $f^{-1}(f(x)) \in f^{-1}(f(E))$. But $f^{-1}(f(x)) = x$ Since f is 1-1. Hence $x \in f^{-1}(f(E))$. Thus

$$(3) \quad E \subseteq f^{-1}(f(E)).$$

From (2) and (3), $E = f^{-1}(f(E))$.

(c) $\implies$ (a) Let $E = \{x\}$, a singleton set. Using the assumption $E = f^{-1}(f(E))$ with this E ,

$$(4) \quad x = f^{-1}(f(x)).$$

Now if f is not 1-1, then there exist x_1 and x_2 , $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. Then $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$. This implies by (4), $x_1 = x_2$ which is a contradiction.

Prove: $x \in f^{-1}(\cup E_\alpha) \implies$ there exists $y \in \cup E_\alpha$ such that $f(x) = y$. Then $y \in E_{\alpha_0}$ for some α_0 . this means that $x \in f^{-1}(E_{\alpha_0})$. Hence $x \in \cup f^{-1}(E_\alpha)$ and thus

$$(5) \quad f^{-1}(\cup E_\alpha) \subseteq \cup f^{-1}(E_\alpha)$$

Conversely, $x \in \cup f^{-1}(E_\alpha)$. Then, $x \in f^{-1}(E_{\alpha_0})$ for some α_0 . This implies that there exists $y \in E_{\alpha_0}$ such that $f(x) = y$. As $y \in E_{\alpha_0} \subseteq \cup E_\alpha$, this show that $x \in f^{-1}(\cup E_\alpha)$ and thus

$$(6) \quad \cup f^{-1}(E_\alpha) \subseteq f^{-1}(\cup E_\alpha)$$

From (5) and (6), we obtain

$$f^{-1}(\cup E_\alpha) = \cup f^{-1}(E_\alpha).$$