

Solutions to Selected Problems of Test 2

2-(1) See examples 3.36 and 3.37

2-(2) See exercise 7, 4.3

3. 1-T, 2-T, 3-T, 4-F, 5-F, 6-F, 7-T, 8-F, 9-T, 10-F

4.

(1) Consult our textbook.

(2) Let $f(x) = e^x - \cos x - 1$. Note that f is continuous on $\mathbb{R}$ and $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) > 0$. Hence by the Intermediate Value Theorem, there exists $x^* \in (0, \frac{\pi}{2})$ such that $f(x^*) = 0$.

(3) Since f is continuously differentiable over $[a, b]$, f' is continuous over $[a, b]$. Hence $|f'(x)| < M$ for some $M > 0$ and for all $x \in [a, b]$. Hence using MVT, for $x, y \in [a, b]$, there exists $c \in (a, b)$ such that $f(x) - f(y) = f'(c)(x - y)$. Hence, given $\epsilon > 0$, choose $\delta = \epsilon/M$ so that for $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |f'(c)(x - y)| \leq M|x - y| < M \cdot \delta = M \cdot \epsilon/M = \epsilon.$$

(4) $\implies$ For this direction, there is nothing to prove, since if f is continuous over $(0, \infty)$, then it is certainly continuous at 1.

$\Leftarrow$ Since f is continuous at 1, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|1 - a| < \delta_1$ implies $|f(1) - f(a)| < \epsilon$. Now for any $x \in (0, \infty)$, choose $\delta = x \cdot \delta_1$, then $|1 - \frac{y}{x}| < \delta_1$, or equivalently $|x - y| < \delta$, we have $|f(x) - f(y)| = |f(\frac{y}{x})| < \epsilon$.