Comment on "Breakdown of the Chapman-Enskog expansion and the anisotropic effect for lattice-Boltzmann models of porous flow" [Phys. Fluids 19, 011702 (2007)]

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In a recent letter,[1] Nie and Martys (N&M) claim that they "*have analytically derived the exact macroscopic governing equations of the lattice-Boltzmann model for the case of simple shear flow in porous media*," and "*find that the effective viscosity in the governing equations is different from the one obtained by the Chapman-Enskog expansion*," thereby they assert a "*breakdown of the Chapman-Enskog expansion.*" In this comment, we would like to point out that the discrepancy between the "effective viscosity" ν_* and the viscosity ν derived from the Chapman-Enskog expansion is merely a numerical artifact due to the lattice Bhatnagar-Gross-Krook (BGK) equation, and this problem has been overcome by the lattice Boltzmann equation (LBE) with multiple-relaxation-time (MRT) models.[2–5]

The MRT-LBE with nine discrete velocities can be written as:[4]

$$\mathbf{f}(\boldsymbol{r}_j + \boldsymbol{c}\delta_t,\, t_n + \delta_t) = \mathbf{f}(\boldsymbol{r}_j,\, t_n) - \mathsf{M}^{-1} \cdot \mathsf{S} \cdot \left[\mathbf{m} - \mathbf{m}^{(\mathrm{eq})}\right] + \delta_t \mathbf{F}(\boldsymbol{r}_j,\, t_n), \tag{1}$$

where the boldface symbols denotes nine-tuple vectors,

$$\mathbf{f}(\boldsymbol{r}_j + \boldsymbol{c}\delta_t,\, t_n + \delta_t) := (f_0(\boldsymbol{r}_j, t_n + \delta_t), f_1(\boldsymbol{r}_j + \boldsymbol{c}_1\delta_t,\, t_n + \delta_t), \ldots, f_8(\boldsymbol{r}_j + \boldsymbol{c}_8\delta_t,\, t_n + \delta_t))^\dagger,$$

$\dagger$ denotes transpose operation, thus $\mathbf{f}(\boldsymbol{r}_j + \boldsymbol{c}\delta_t,\, t_n + \delta_t)$ and $\mathbf{f}(\boldsymbol{r}_j,\, t_n)$ are the vectors of the advected and pre-collision distribution functions, respectively; $\mathbf{m}$ and $\mathbf{m}^{(\mathrm{eq})}$ are the vectors of moments and their equilibria, respectively; and $\mathbf{F}$ represents a forcing term. The invertible transform matrix M maps $\mathbf{f}$ to $\mathbf{m}$:[4]

$$\mathbf{m} = \mathsf{M} \cdot \mathbf{f}, \quad \mathbf{f} = \mathsf{M}^{-1} \cdot \mathbf{m}.$$

The relaxation matrix S is diagonal, and its diagonal elements are relaxation rates $0 < s_i < 2$, $i \in \{0,\, 1,\, \ldots, 8\}$. For the D2Q9 model, there are nine moments:[4]

$$\mathbf{m} = (\rho, e, \varepsilon, j_x, q_x, j_y, q_y, p_{xx}, p_{xy})^\dagger,$$

among these moments ρ is the density and $\boldsymbol{j} := (j_x, j_y)$ is the momentum; ρ, e, ε, p_{xx}, and p_{xy} are even-order moments, and j_x, j_y, q_x, and q_y are odd-order ones. In the two-relaxation-time (TRT) model,[5,6] there are only two relaxation rates, s^+ and s^-, for the even-order and odd-order moments, respectively.

In Cartesian coordinates (x, y), the discrete velocities are $\boldsymbol{c}_0 = (0, 0)$, $\boldsymbol{c}_{1,3} = (\pm 1, 0)c$, $\boldsymbol{c}_{2,4} = (0, \pm 1)c$, and $\boldsymbol{c}_{5,6,7,8} = (\pm 1, \pm 1)c$, where $c = \delta_x/\delta_t$. For the Brinkman flow, the forcing term is:

$$\boldsymbol{F} = -\frac{\phi\nu}{k}\boldsymbol{j}, \tag{2}$$

where ϕ is the porosity and k is the permeability. Ginzburg[6] considered the Brinkman flow in the coordinates (η, ζ) rotated with an angle θ with respect to the x-axis: $\eta = x\cos\theta + y\sin\theta$ and $\zeta = -x\sin\theta + y\cos\theta$, and the forcing $\boldsymbol{F} \propto \boldsymbol{u}$ is along the η-axis. The TRT-LBE solution for the streamwise momentum j_η as a function of spanwise coordinate ζ satisfies the following difference equation:[6]

$$\Delta^2_{\zeta,q} j_\eta(\zeta) = \frac{\nu}{\nu_*(1+\delta_\nu)}\frac{\phi}{k} j_\eta(\zeta), \tag{3a}$$

$$\Delta^2_{\zeta,q} f(\boldsymbol{r}_j) = \frac{f(\boldsymbol{r}_j + \delta_t\boldsymbol{c}_q) - 2f(\boldsymbol{r}_j) + f(\boldsymbol{r}_j - \delta_t\boldsymbol{c}_q)}{\Theta_q^2}, \tag{3b}$$

$$\delta_\nu = \left[\Theta_q^2\left(\Lambda - \frac{1}{4}\right) - \frac{1}{3}\Lambda\right]\frac{\phi\nu}{k\nu_*}, \tag{3c}$$

$$\Lambda = \left(\frac{1}{s^+} - \frac{1}{2}\right)\left(\frac{1}{s^-} - \frac{1}{2}\right), \tag{3d}$$

where $\Theta_q = \delta_t\|\boldsymbol{c}_q \cdot \hat{\boldsymbol{\zeta}}\|$, $\nu = (1/s^+ - 1/2)/3$, and $\delta_\nu = (\nu/\nu_* - 1)$.

For the lattice BGK model, $s^+ = s^- = 1/\tau$, and $\nu = (\tau - 1/2)/3$. Therefore for the cases $\theta = 0$ ($\Theta^2 = 1$) and $\theta = \pi/4$ ($\Theta^2 = 1/2$), Eq. (3c) leads to

$$\delta_\nu = \frac{(8\tau^2 - 8\tau - 1)}{12}\frac{\phi}{k}\frac{\nu}{\nu_*}, \quad \theta = 0, \tag{4a}$$

$$\delta_\nu = \frac{(2\tau^2 - 2\tau - 1)}{12}\frac{\phi}{k}\frac{\nu}{\nu_*}, \quad \theta = \frac{\pi}{4}, \tag{4b}$$

which correspond to Eqs. (11) and (12) in N&M,[1] respectively. Since ν_* depends on the flow direction with respect to the underlying lattice, hence ν_* is *anisotropic.* Furthermore, because $\nu_* \neq \nu$ in general, N&M declare a *breakdown* of the Chapman-Enskog expansion. However, if the TRT-LBE model is used, $\nu_* = \nu$ if

$$\left(3\Theta^2 - 1\right)\Lambda = \frac{3}{4}, \tag{5}$$

which can be easily satisfied by setting $s^- = 4(2 - s^+)/(4 + s^+)$ for $\theta = 0$ and $s^- = (2 - s^+)/(1 + s^+)$ for $\theta = \pi/4$. This is impossible for the LBGK model to achieve because it has only one adjustable parameter τ.

Ginzburg's work[6] pin-points the errors in N&M.[1] First of all, the theoretical viscosity ν used by N&M was derived with a *constant* forcing. In other words, the Chapman-Enskog analysis for the Brinkman flow with a space dependent forcing term was never carried out in N&M. Secondly, the MRT-LBE model can eliminate the discrepancy of $\nu_* \neq \nu$ and the anisotropy of ν_*. That is, both problems are merely the numerical artifacts of the LBGK model. They have nothing to do with the Chapman-Enskog expansion *per se.* N&M misinterpret the defects of the lattice BGK model as a breakdown of the Chapman-Enskog expansion.

It is interesting to note that the Poiseuille flow is another case to demonstrate deffects of the LBGK model with the bounce-back boundary conditions. The LBGK analytic solution for the Poiseuille flow exhibits the same discrepancy $\nu_* \neq \nu$ and the anisotropy of ν_*.[7–10] However, there is a salient difference between the Brinkman flow and the Poiseuille flow, the error of the LBGK model comes from the bulk in the former case, while it comes from the boundary conditions in the latter.

N&M also speculate that finite Knudsen number may be a cause for the breakdown of the Chapman-Enskog expansion. However, the so-called Knudsen number $\mathrm{Kn} \propto (\tau - 1/2)/L \propto$

ν/L in the LBGK equation is misleading, for the LBGK equation is only valid for near incompressible flows and does not converge to the Boltzmann equation in the limit of δ_x and δ_t go to zero.[11] Therefore it is futile to expect the LBGK equation to capture the Knudsen layer in rarefied gas.

In conclusion, we have unequivocally shown that the assertions by N&M that the effective viscosity ν_* is anisotropic and differs from ν derived from the Chapman-Enskog analysis, and that thus the Chapman-Enskog expansion breaks down for the lattice Boltzmann equation are both erroneous. The observations of N&M are due to numerical artifacts of the lattice BGK model, which have been mischaracterized as viscous effects by N&M. These problems can be effectively solved by using either TRT or MRT models.[2–6]

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