

ASYMPTOTIC BEHAVIOR OF THE CHRISTOFFEL FUNCTION
RELATED TO A CERTAIN UNBOUNDED SET

L.S. Luo and J. Nuttall

Department of Physics

The University of Western Ontario

London, Ontario, Canada. N6A 3K7

ABSTRACT. We study the asymptotic behavior, as the degree approaches infinity, of the Christoffel function at a fixed point z corresponding to a weight function of the type $\exp(-|z|^\lambda)$ on the set $|\arg z| = \frac{\pi}{2} + \alpha$. The method generalizes that of Rakhmanov and also Mhaskar and Saff.

1. Introduction

In Quantum Mechanics there is a class of problems for which the required energy $E(\beta)$ may formally be expressed in terms of a parameter β by means of a power series with zero radius of convergence. Examples include the anharmonic oscillator and the hydrogen atom Stark effect. We have shown [2] that these series may be summed with the help of polynomials $Q_n(z)$ extremal with respect to the weight $\rho(z)\exp(-|z|^\lambda)$, $\lambda \geq 1$, on the set $\Gamma(\alpha)$, where, with $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$,

$$(1.1) \quad \Gamma(\alpha) = \{z \in \mathbb{C}: |\arg z| = \frac{\pi}{2} + \alpha\}.$$

We assume that $\rho(z)$ is a real positive locally integrable function for $z \in \Gamma(\alpha)$, that $\log \rho(z)$ is also locally integrable, and that

$$(1.2) \quad \lim_{|z| \rightarrow \infty} |z|^{-\lambda} \log \rho(z) = 0.$$

We are here principally concerned with the quantity $K_n(z)$.

$$(1.3) \quad K_n(z) = \frac{\sup_{\deg P_n \leq n} |P_n(z)|^2}{\int_{\Gamma(\alpha)} |P_n(y)|^2 \rho(y) \exp(-|y|^\lambda) |dy|}.$$

$Q_n(z)$ is the polynomial (unique up to a multiplicative constant factor) for which the sup is attained. If $H_n(z)$ is the orthonormal polynomial of degree n corresponding to the weight, i.e.

$$(1.4) \quad \int_{\Gamma(\alpha)} H_n(z) \overline{H_m(z)} \rho(z) \exp(-|z|^\lambda) |dz| = \delta_{nm},$$

then

$$(1.5) \quad K_n(z) = \sum_{j=0}^n |H_j(z)|^2.$$

Thus $K_{n-1}(z)$ is the reciprocal of the Christoffel function as defined by Nevai [7] (see also Szegő [9]).

Here we analyze the asymptotics of $K_n(z)$ as $n \rightarrow \infty$ for fixed z with methods that closely follow those of Rakhmanov [8] and to a lesser extent Mhaskar and Saff [3,4]. We refer the reader to Rakhmanov's paper at many places where the argument is the same or is but slightly modified.

Our result is presented in the form of Theorem 4.5.

2. Minimum problem in potential theory

Suppose M_n is the set of all positive measures σ such that $\text{supp } \sigma \subset \Gamma(\alpha)$ and $\int_{\Gamma(\alpha)} d\sigma(t) = n$, and define the logarithmic potential

$$(2.1) \quad V_\sigma(z) = - \int \log |z-t| d\sigma(t).$$

With $V_0(z) = |z|^\lambda$, $z \in \Gamma(\alpha)$, we define

$$(2.2) \quad J(\sigma) = \int (V_\sigma(t) + V_0(t)) d\sigma(t).$$

Mhaskar and Saff [4] have discussed the problem of minimizing $J(\sigma)$ for $\sigma \in \mathbb{M}_n$ in the case when $\Gamma(\alpha)$ is replaced by the real line. As far as parts (a) to (f) of their Theorem 2.3 are concerned, their proofs apply with trivial modifications to our case $\Gamma(\alpha)$. In terms of the notation used here, patterned after Rakhmanov [8], we may therefore assert that there exists a unique measure $\nu \in \mathbb{M}_n$ with compact support and of non-zero capacity such that

$$(2.3) \quad J(\nu) = j = \inf_{\sigma \in \mathbb{M}_n} J(\sigma) > -\infty.$$

If we define

$$(2.4) \quad m = 2j - \int V_0(z) d\nu(z)$$

then

$$(2.5) \quad 2V_\nu(z) + V_0(z) \geq m, \quad z \in \Gamma(\alpha), \quad \text{except possibly for a set of capacity zero,}$$

and

$$(2.6) \quad 2V_\nu(z) + V_0(z) \leq m, \quad z \in \text{supp } \nu.$$

From this information we have

Lemma 2.1

$$(2.7) \quad (i) \quad \text{supp } \nu = \Delta(R) = \{z \in \Gamma(\alpha) : |z| \leq R\}, \quad \text{for some real positive } R.$$

$$(2.8) \quad (ii) \quad 2V_\nu(z) + V_0(z) > m, \quad z \in \Gamma(\alpha) \setminus \Delta(R).$$

Proof (i). The measure ν must be symmetric about the real axis for otherwise reflection in the real axis would lead to another measure with the same value of J , which would contradict uniqueness. The function $f(r)$ given by

$$(2.9) \quad f(r) = 2V_\nu(re^{i\alpha}) + V_0(re^{i\alpha}), \quad 0 \leq r < \infty,$$

is convex downwards, which may be seen by checking that

$$(2.10) \quad -\log|re^{i\alpha}-se^{i\alpha}| - \log|re^{i\alpha}-se^{-i\alpha}|, \quad 0 \leq r < \infty$$

is convex in r . The arguments of Rakhmanov [8] and Mhaskar and Saff [4] then show that $\text{supp } v$ must be connected so that (i) follows by symmetry.

(ii) If there were $r_0 > R$ for which $f(r_0) \leq m$, then convexity shows that $f(r) < m$, $R < r < r_0$, which contradicts (2.5). $\square$

3. Boundary value problem

We now outline the use of standard methods to find a real function $\tilde{V}(z)$, harmonic for $z \in \mathbb{C} \setminus \Lambda(R)$, and continuous for $z \in \mathbb{C}$, which satisfies the conditions

$$(3.1) \quad \tilde{V}(z) = 1/2(m - V_0(z)), \quad z \in \Lambda(R)$$

$$(3.2) \quad \tilde{V}(z) = -n \log|z| + O(|z|^{-1}), \quad |z| \rightarrow \infty.$$

With $\eta = 2\alpha/\pi$ it may be shown that there is a unique γ , $0 < \gamma < \pi$ for which

$$(3.3) \quad \int_{-1}^1 \frac{(t-c)(t-c^{-1})}{t^2} \left[\frac{t-1}{t+1} \right]^\eta dt = 0,$$

where the path of integration lies in the upper half-plane, and $c = e^{i\gamma}$. This can be seen by regarding (3.3) as an equation for $\cos \gamma$. We set

$$(3.4) \quad z_0 = \int_1^c \frac{(t-c)(t-c^{-1})}{t^2} \left[\frac{t-1}{t+1} \right]^\eta dt$$

where the path of integration is on the unit circle in the upper half-plane, and define

$$(3.5) \quad Z(w) = |z_0|^{-1} \int_1^w \frac{(t-c)(t-c^{-1})}{t^2} \left[\frac{t-1}{t+1} \right]^\eta dt, \quad |w| \geq 1$$

with integration path in $|t| \geq 1$.

Lemma 3.1 The transformation $w \rightarrow z = Z(w)$ conformally maps $|w| > 1$ onto $z \in \mathbb{C} \setminus \Lambda(1)$.

Proof. This is similar to a Schwarz transformation. [5, p.363]. $\square$

Now write

$$(3.6) \quad g(w) = -1/2 |Z(w)|^\lambda, \quad |w| = 1$$

and define

$$(3.7) \quad Q(w) = -(i\pi)^{-1} \int_{|t|=1} g(t)(t-w)^{-1} dt + Q_0$$

where

$$(3.8) \quad Q_0 = (2\pi i)^{-1} \int_{|t|=1} g(t)t^{-1} dt.$$

Then $Q(w)$ is analytic in $|w| > 1$ and satisfies [6, p.107]

$$(3.9) \quad \operatorname{Re} Q(w) = g(w), \quad |w| = 1$$

where $Q(w)$ means the limit from $|w| > 1$.

Lemma 3.2 The solution of (3.1), (3.2) is

$$(3.10) \quad \tilde{V}(z) = \operatorname{Re}(R^\lambda Q(w) - n \log w + 1/2 m), \quad z = R Z(w), \quad |w| \geq 1,$$

if and only if

$$(3.11) \quad m = 2[n \log(|z_0|/R) - R^\lambda Q_0].$$

Proof. The function given is harmonic and satisfies (3.1). Since $Z(w) \sim |z_0|^{-1} w$ as $|w| \rightarrow \infty$, (3.2) will hold if (3.11) is true [1].

It must be possible to represent $\tilde{V}(z)$ as $V_\mu(z)$ for some signed measure μ with $\operatorname{supp} \mu \subset \Lambda$. On $\Lambda(R)$, $V_\mu(z) = V_\nu(z)$ except possibly for a set of capacity zero, and since $\Lambda(R)$ has no irregular points, the uniqueness theorem shows that $\mu \equiv \nu$ [1, p.245], i.e.

$$V_v(z) \equiv \tilde{V}(z). \quad \square$$

Now in the vicinity of the point c , for $|w| = 1$, $\arg Z(w) = \frac{\pi}{2} + \alpha$, so that we may write $|Z(w)| = Z(w)\exp(-i(\frac{\pi}{2} + \alpha))$, which shows that $g(w)$ has an analytic continuation in a neighborhood of $w = c$, as does $Q(w)$. We also have, with $B = be^{-2i\gamma}$, b real and positive,

$$(3.12) \quad Z(w) = e^{i(\frac{\pi}{2} + \alpha)} (1 + B(w-c)^2) + O(|w-c|^3), \quad w \rightarrow c,$$

so that

$$\begin{aligned} \tilde{V}(z) - \tilde{V}(RZ(c)) &= \operatorname{Re}[(R^\lambda Q'(c) - n/c)(w-c) + O(|w-c|^2)], \quad w \rightarrow c \\ (3.13) \quad &= \operatorname{Re}\left[e^{i(\frac{\pi}{4} - \frac{\alpha}{2})} (BR)^{-1/2} (R^\lambda Q'(c) - n/c) (z - RZ(c))^{1/2}\right] \\ &+ o(|z - RZ(c)|^{1/2}), \\ &z \rightarrow RZ(c). \end{aligned}$$

As in [8], we argue that the convexity of $V_v(z)$ outside $\Delta(R)$ and the need to satisfy (2.8) mean that

$$(3.14) \quad R^\lambda = n/A, \quad A = cQ'(c).$$

Of course, this quantity must be real and positive, and we have found R .

Lemma 3.3 It is possible to write $dv(z) = v'(z)|dz|$, $z \in \Delta(R)$, with

$$(3.15) \quad v'(z) = \frac{n}{R} a\left[\frac{z}{R}\right], \quad z \in \Delta(R).$$

The function $|z|^{\frac{\eta}{1-\eta}} a(z)$ is continuous, $z \in \Delta(1)$.

Proof. The function $v'(z)$ is proportional [6] to the discontinuity of $\int_{\Delta(R)} (t-z)^{-1} dv(t)$ across $\Delta(R)$. With (3.10) this may be related

to $Q(w)$, which is analytic everywhere on $|w| = 1$ except at $w = \pm 1$. $\square$

4. Bounds on $K_n(z)$

Following Rakhmanov [8], we first obtain in Lemma 4.3 bounds in the case $\rho \equiv 1$. The lower bound is constructed by evaluating (1.3) for a particular polynomial $P(z)$ of degree n , related to the measure ν^* defined below.

For x, y with $\arg x = \arg y = \frac{\pi}{2} + \alpha$, $|x| < |y|$, define

$$(4.1) \quad \delta(x, y) = \{z \in \mathbb{C}: z = re^{i(\frac{\pi}{2} + \alpha)}, |x| \leq r \leq |y|\}.$$

Suppose that n is odd; the argument is similar for n even. Choose points z_j , $j = 1, \dots, (n+1)/2$ such that $\arg z_j = \frac{\pi}{2} + \alpha$, $|z_1| > |z_2| > \dots > |z_{(n+1)/2}| = 0$, and

$$\int_{\delta(z_1, RZ(c))} d\nu(t) = 1/2,$$

(4.2)

$$\int_{\delta(z_{k+1}, z_k)} d\nu(t) = 1, \quad k = 1, \dots, (n-1)/2.$$

(Note that the symmetry of ν ensures that $z_{(n+1)/2} = 0$.) We define $z_k = \bar{z}_{n-k+1}$, $k = (n+1)/2, \dots, n$.

Define the measure ν^* as

$$(4.3) \quad \nu^* = \sum_{j=1}^n \delta(z_j)$$

and

$$(4.4) \quad V^*(z) = V_{\nu^*}(z) = - \sum_{j=1}^n \log |z - z_j|.$$

Lemma 4.1

$$(4.5) \quad (i) \quad V_v(z) - V^*(z) \leq \left[\frac{1-\eta}{1-2\eta} \right] \log Cn, \quad z \in \mathbb{C},$$

for some positive constant C .

$$(4.6) \quad (ii) \quad V_v(z) - V^*(z) \geq -\log \left[1 + R/D(z, \Delta(R)) \right], \quad z \in \mathbb{C} \setminus \Delta(R),$$

for fixed z and sufficiently large R . (The quantity $D(z, \Delta(R))$ means the distance from z to $\Delta(R)$).

Proof. (i) Suppose $x \in \Delta(R)$ lies in the interior of the interval $\delta(z_{k+1}, z_k)$, $k \leq (n-1)/2$. Then just as [8, p.171] we deduce that

$$(4.7) \quad V(x) - V^*(x) \leq \log 2R - \int_{\delta(z_{k+1}, z_k)} \log |x-t| \, dv(t).$$

We obtain the upper bound taking $k = (n-1)/2$, $x = 0$, and using Lemma 3.3.

(ii) The proof is as in [8, p.171]. $\square$

One more piece of information is required before we can compute the lower bound. Define $W(z)$ by

$$(4.8) \quad W(z) = \exp(m - 2V_v(z)).$$

Then we have

Lemma 4.2

$$(4.9) \quad \lim_{n \rightarrow \infty} \frac{1}{2R} \int_{\Gamma(\alpha)} W(z) \exp(-|z|^\lambda) |dz| = 1.$$

Proof. Since $W(z) \exp[-|z|^\lambda] = 1$, $z \in \Delta(R)$, we have

$$(4.10) \quad \int_{\Delta(R)} W(z) \exp(-|z|^\lambda) |dz| = 2R.$$

For $z \in \Gamma(\alpha) \setminus \Delta(R)$, we write

$$W(z) \exp(-|z|^\lambda) = \exp[2n\{\operatorname{Re}(\log w - Q(w)/A) - 1/2 |Z(w)|^\lambda/A\}],$$

$$z = RZ(w).$$

$$(4.11) \quad = \exp(2nh(w)) \quad \text{say.}$$

On account of Lemma 2.1 (ii) we know that $h(w) < 0$ for $z \in \Gamma(\alpha) \setminus \Delta(R)$. Thus $\int_{\Gamma(\alpha) \setminus \Delta(R)} W(z) \exp(-|z|^\lambda) |dz|$ can be approximated for large n by the contribution from those z near $\Delta(R)$. For z near $RZ(c)$, i.e. w near c , we have

$$(4.12) \quad h(w) = \operatorname{Re}[C_1(w-c)^2] + O(|w-c|^3)$$

for some constant C_1 . We conclude that

$$(4.13) \quad \int_{\Gamma(\alpha) \setminus \Delta(R)} W(z) \exp(-|z|^\lambda) |dz| = R(\operatorname{const.} n^{-1} + O(n^{-2}))$$

and the Lemma follows. $\square$

Lemma 4.3 In the case $\rho \equiv 1$, for large n , and z in a fixed compact subset of $\mathbb{C} \setminus \Delta(R)$,

$$(4.14) \quad |w| W(z) / \left[\pi D(z, \Delta(R)) \right] \geq K_n(z) \\ \geq C n^{-(3/\lambda + (2-2\eta)/(1-2\eta))} W(z),$$

$$z = RZ(w).$$

Proof. For the lower bound as [8, p.172] we use the polynomial

$$P(z) = e^{m/2} \prod_{i=1}^n (z - z_i), \quad \text{so that}$$

$$(4.15) \quad |P(z)| = \exp[1/2m - V^*(z)]$$

and

$$(4.16) \quad |P(z)|^2 = W(z) \exp[2(V_v(z) - V^*(z))].$$

We use Lemmas 4.1 and 4.2 to estimate

$$(4.17) \quad \frac{|P(z)|^2}{\int_{\Gamma(\alpha)} |P(z)|^2 \exp(-|z|^\lambda) |dz|}.$$

The proof of the upper bound is as in [8, p.160]. $\square$

Lemma 4.4 As $n \rightarrow \infty$ for z fixed, $|\arg z| \leq \frac{\pi}{2}$, $|z| > 0$, we have

$$(4.18) \quad \log W(z) = C_3 n^{1 - \frac{1}{\lambda(\eta+1)}} \left[\operatorname{Re} z^{\frac{1}{\eta+1}} \right] (1 - A^{-1} Q'(1)) (1 + o(1))$$

where

$$(4.19) \quad C_3 = 2A^{-\frac{1}{\lambda}} \left[|z_0|^{-1} 2^{-\eta(\eta+1)^{-1}} (1-c)(1-c^{-1}) \right]^{-\frac{1}{\eta+1}}.$$

Proof. We have from (3.10), (4.8)

$$(4.20) \quad \log W(z) = 2n \operatorname{Re} [\log w - A^{-1} Q(w)], \quad z/R = Z(w).$$

Near $w = 1$, (3.5) gives

$$(4.21) \quad Z(w) = C_2 (w-1)^{\eta+1} (1 + O(|w-1|)),$$

where

$$(4.22) \quad C_2 = |z_0|^{-1} 2^{-\eta(\eta+1)^{-1}} (1-c)(1-c^{-1}),$$

so that, since $R \rightarrow \infty$,

$$(4.23) \quad w-1 = (z/(RC_2))^{\frac{1}{\eta+1}} (1 + O(|z/R|)).$$

Writing

$$(4.24) \quad \log w - A^{-1} Q(w) = (1 - A^{-1} Q'(1))(w-1) + o(|w-1|)$$

leads to the result. $\square$

Theorem 4.5 Suppose $\rho(z)$, $z \in \Gamma(\alpha)$, is real, positive and integrable and $\log \rho(z)$ is locally integrable, and

$$(4.25) \quad \lim_{|z| \rightarrow \infty} |z|^{-\lambda} \log \rho(z) = 0, \quad z \in \Gamma(\alpha).$$

With $K_n(z)$ defined by (1.3),

$$(4.26) \quad \lim_{n \rightarrow \infty} \frac{\log K_n(z)}{n^{1 - \frac{1}{\lambda(\eta+1)}}} = C_3(1 - A^{-1}Q'(1)) \operatorname{Re} \left[z^{\frac{1}{\eta+1}} \right]$$

for fixed z , $|\arg z| \leq \alpha$, $|z| > 0$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $\lambda > 1$. For $\lambda = 1$, the result holds for $0 < \alpha < \frac{\pi}{2}$.

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Proof. For $\rho(z) \equiv 1$ we use (4.18) in (4.14). The case $\rho(z) \not\equiv 1$ follows as in [8, Sec. 4].

We note that the above result in the case $\alpha = 0$, $\lambda > 1$, agrees with that of Rakhmanov [8].

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