The ordering of critical periodic points in coordinate space by symbolic dynamics

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In this paper, we study the symbolic dynamics equipped with the DGP product "$*$" and the shift operator φ conjugate to the unimodal mapping f_μ: $I \mapsto I$ over the interval $I = [-1, 1]$. We prove two theorems, and give an explicit formula for the rule of ordering the points on the n-tupling periodic trajectory $\{\varphi^k(\sigma^{*n}(p+1))\}$ in coordinate space with arbitrarily large p and n.

Until recently, to calculate the fractal dimension of critical periodic points of unimodal maps with significant large period has been a difficult task, because the computing time for the ordering of the points on the n-tupling periodic trajectory increases exponentially with n. Such a calculation is justified because it is believed that there may exist a new global regularity of fractal dimensions on critical points [1]. Moreover, the thermodynamical formalism of dimension of complicated objects has received much attention [2–7]. In this formalism, one of the important and essential aspects is how to calculate the partition function of a complex system by using the ordering of points on trajectories in the coordinate space. Even though this can be done numerically for trajectories with relatively short period, their analytical ordering presents serious difficulties as shown in ref. [1]. First of all, the sequence of coordinate points for an arbitrary trajectory is an infinite one in general. Second, each coordinate point on a trajectory is represented by an infinite sequence of "symbols" in general, as explained later.

In this work, we obtain an analytic method to reveal the rule of the ordering of the points on arbitrary n-tupling periodic trajectories of unimodal mapping in the coordinate space. Our work in this paper is an extension of a work of Feigenbaum [2] where the rule of ordering points on the period-doubling trajectories, i.e., trajectories with period 2^n, $n=0, 1, 2, \ldots$, has been derived. We extend the result to trajectories with period k^n for arbitrary $k=2, 3, \ldots$, and $n=1, 2, \ldots$, and provide rigorous proofs. Our approach in this work involves symbolic dynamics and some elementary results in number theory.

Consider a unimodal mapping [8] f_μ: $I \mapsto I$ over the interval $I = [-1, 1]$ depending on a parameter μ. The location of the unique maximum of f_μ can be normalized to be the origin. For an arbitrary point $x_0 \in I$, the set $O_f(x_0) = \{f^n_\mu(x_0)\} \equiv \{x_n\}$ is called a trajectory with initial point x_0. The symbolic dynamics can be easily introduced by discriminating among the three possible cases $x_n < 0$, $x_n = 0$ and $x_n > 0$. Therefore, each trajectory can be associated with an infinite sequence of three symbols, L, C and R:

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_i \ldots , \qquad (1)$$

where $\sigma_i \in \{L, C, R\}$ is determined by the following rule:

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$$\begin{aligned}\sigma_i &= \mathrm{L}\,, \quad \text{if } x_i<0\,,\\ &= \mathrm{C}\,, \quad \text{if } x_i=0\,,\\ &= \mathrm{R}\,, \quad \text{if } x_i>0\,. \end{aligned} \tag{2}$$

This sequence σ is referred to as a word, or an itinerary of the corresponding trajectory. All words with s symbols form a symbolic space Σ_s, in particular, $\Sigma_2=\{\sigma=\prod_{i=1}^{\infty}\sigma_i\,|\,\sigma_i\in\{\mathrm{L},\mathrm{R}\}\}$. A periodic trajectory of f_μ with period p corresponds to a periodic word $\sigma(p)$ with the same period, i.e.,

$$\sigma(p)=\sigma_1\sigma_2...\sigma_{p-1}\sigma_p\sigma_1\sigma_2...\sigma_{p-1}\sigma_p...\,. \tag{3}$$

Since $\sigma(p)$ has a repeated pattern with p bits, one can represent a periodic word $\sigma(p)$ by the p-bit sequence

$$\sigma(p)=\sigma_1\sigma_2...\sigma_{p-1}\sigma_p \tag{4}$$

without losing any information. Thus, a periodic word is sometimes referred to as a finite word. For any non-periodic word, we adopt the notation $\sigma=\sigma(\infty)$.

It is obvious that the symbol C can only appear in some periodic words at most once within one period. A periodic word with a symbol C is referred to as a superstable word [8]. We will denote a superstable word by $\sigma(p+1)\equiv\sigma(p)\mathrm{C}$ in such a way that the symbol C is always placed at the end of the word $\sigma(p+1)$. It is also a convention that superstable words also belong to Σ_2.

The dynamics of a unimodal mapping $f_\mu:\mathrm{I}\mapsto\mathrm{I}$ is closely related to the dynamics of a shift operator $\varphi:\Sigma_2\mapsto\Sigma_2$. Roughly speaking, mapping f_μ and shift φ are topologically conjugate [9,10]. Therefore, to study the dynamics of a unimodal mapping $f_\mu:\mathrm{I}\mapsto\mathrm{I}$ is equivalent to studying the dynamics of the conjugate shift $\varphi:\Sigma_2\mapsto\Sigma_2$ with

$$\varphi(\sigma)=\sigma_2\sigma_3...\sigma_i...\,. \tag{5}$$

A periodic word remains periodic under the shift operation:

$$\varphi(\sigma(p))=\sigma_2\sigma_3...\sigma_p\sigma_1\sigma_2\sigma_3...\sigma_p\sigma_1...\,. \tag{6}$$

Therefore, it can be rewritten as

$$\varphi(\sigma(p))=\sigma_2\sigma_3...\sigma_p\sigma_1\,. \tag{7}$$

Now it is obvious that the shift operator φ becomes a cyclic shift operator for periodic words. We also define a truncation operator $\bar{\varphi}$ for a word σ:

$$\bar{\varphi}^k(\sigma)=\sigma_1\sigma_2...\sigma_{k-1}\sigma_k\,, \tag{8}$$

with the convention that $\bar{\varphi}^0(\sigma)=\varnothing$.

The ordering on a trajectory $\{x_n\}$ in coordinate space is naturally implied by the ordering of the real numbers. We introduce a corresponding ordering between words in the space Σ_2 – the Metropolis–Stein–Stein (MSS) order [8,11]. However, the MSS order between words corresponds to the ordering of real numbers on the parameter space of the mapping f_μ [11]. But, if one looks at an inverse periodic word

$$\bar{\sigma}(p)=\sigma_p\sigma_{p-1}...\sigma_2\sigma_1\,, \tag{9}$$

the MSS order is actually the order in coordinate space of mapping f_μ [12]. Therefore, for all periodic words $\sigma(p)$, the MSS order also reflects the ordering between words on coordinate space, because the mapping between $\sigma(p)$ and $\bar{\sigma}(p)$ is one-to-one and onto, and $\bar{\sigma}(p)$ needs not to be mentioned at all later.

In order to make an ordering, a reference point is needed. Such a reference point, with respect to a given periodic trajectory, is the maximal word defined as follows: if there exists an integer $r<p$ and

$$\sigma_{\mathrm{M}}(p)=\varphi^r(\sigma(p)) \tag{10}$$

for a given word $\sigma(p)$ such that

$$\sigma_M(p) \succ \varphi^k(\sigma(p)) \quad \forall k \in \{0, 1, 2, ..., r-1, r+1, ..., p-1\} , \tag{11}$$

then the word $\sigma_M(p)$ is called the maximal word on the trajectory $\{\varphi^k(\sigma(p))\}$. Where the ordering $\succ$ is in the sense of the MSS order. In a similar fashion, we can define a minimal word $\sigma_m(p)$.

It is our objective to study periodic words in the space Σ_2. It is thus important to obtain bifurcation periodic words, which have finite bits. For this reason, we must introduce a composition rule between finite words, which is the Derrida, Gervois and Pomeau (DGP) product denoted by "$*$" [13]. To give a formal definition of the DGP product, we need some useful auxiliary functions. For any integer $d > 1$ and n, one can always represent n in mode (d):

$$n = m_r d^r + m_{r-1} d^{r-1} + ... + m_i d^i + ... + m_1 d + m_0 . \tag{12}$$

Let us define the functions

$$\mathscr{P}_d^s(n) = m_s \tag{13}$$

and

$$\mathscr{Q}_d^s(n) = n \bmod (d^s) , \tag{14}$$

with the notation $\mathscr{Q}_d(n) \equiv \mathscr{Q}_d^1(n)$.

The R-parity number of a word $\sigma(p)$, $J(\sigma(p))$, is the number of appearances of the symbol R in the word $\sigma(p)$. The R-parity $\delta(\sigma(p))$ is defined as

$$\begin{aligned} \delta(\sigma(p)) &= 0 , \quad \text{if } J(\sigma(p)) \text{ is even} , \\ &= 1 , \quad \text{if } J(\sigma(p)) \text{ is odd} , \end{aligned} \tag{15}$$

i.e.,

$$\delta(\sigma(p)) = \tfrac{1}{2}[1 - (-1)^{J(\sigma(p))}] , \tag{16}$$

with the convention that $\delta(\varnothing) = 0$. The parity inverse operator t reverses symbols L, C and R in the following way:

$$\mathrm{L}^t = \mathrm{R} , \quad \mathrm{C}^t = \mathrm{C} , \quad \mathrm{R}^t = \mathrm{L} . \tag{17}$$

It is understood that $\sigma^t = \sigma_1^t \sigma_2^t ... \sigma_i^t ...$. We can also make the operator t depend on a word $\sigma(p)$ such as

$$t(\sigma(p)) \equiv t^{\delta(\sigma(p))} , \tag{18}$$

that is

$$\begin{aligned} t(\sigma(p)) &= t , \quad \text{if } \delta(\sigma(p)) = 1 . \\ &= 1 , \quad \text{if } \delta(\sigma(p)) = 0 , \end{aligned} \tag{19}$$

where 1 denotes the identity operator.

The definition of the DGP product between two superstable words $\sigma(p+1) \equiv \sigma(p)\mathrm{C}$ and $\omega(q+1) \equiv \omega(q)\mathrm{C}$ is

$$\rho(r+1) = \sigma(p+1) * \omega(q+1) = \rho_1 \rho_2 ... \rho_{r-1} \rho_r \mathrm{C} , \tag{20}$$

where $r+1 = (p+1)(q+1)$ and

$$\begin{aligned} \rho_i &= \sigma_j , && \text{if } \mathcal{Q}_{p+1}(i) = j \neq 0 , \\ &= \omega_k^{t(\sigma(p))} , && \text{if } \mathcal{Q}_{p+1}(i) = 0 \text{ and } \mathcal{P}^1_{p+1}(i) = k , \\ &= \mathrm{C} , && \text{if } i = (p+1)(q+1) . \end{aligned} \tag{21}$$

We should mention some important properties of the DGP product. First, the product is not commutative; second, it is associative so that the power of any (superstable) periodic word $\sigma(p+1)$, denoted as $\sigma^{*k}(p+1)$, $k=2, 3, \ldots$, is well defined; third, for any maximal words $\sigma_{\mathrm{M}}(p+1)$ and $\omega_{\mathrm{M}}(q+1)$, their product is preserved as a maximal word.

We also need to use the order inverse operator T [12] and the invariant coordinate Θ [14]. Given an ordered sequence of words: ω, κ, ..., ρ, η, then

$$(\omega \succ \kappa \succ \ldots \rho \succ \eta)^T = \eta \succ \rho \succ \ldots \kappa \succ \omega . \tag{22}$$

The operator T can depend on a word $\sigma(p)$ similarly as the parity inverse operator t does, i.e.,

$$T(\sigma(p)) \equiv T^{\delta(\sigma(p))} . \tag{23}$$

The lth invariant coordinate of a word σ is defined as the following:

$$\Theta(\bar{\varphi}^l(\sigma)) = (-1)^{J(\bar{\varphi}^l(\sigma))} . \tag{24}$$

The lth invariant coordinate $\Theta(\bar{\varphi}^l(\sigma))$ is sometimes abbreviated as $\Theta(l)$ when no confusion can arise.

From hereafter, we shall only consider periodic words in Σ_2, which form a subspace in Σ_2 denoted as Σ_2'. To avoid confusion, a word σ means a periodic word unless declared otherwise hereafter. Also, we will insist on using the notation $\sigma(p)$ and $\sigma(p+1)$ to denote a periodic word with period p and a superstable periodic word with period $p+1$, respectively.

Lemma 1. The parity of any power of a (superstable) word $\sigma \equiv \sigma(p+1)$ is determined by

$$\delta(\sigma^{*n}) = \delta(\sigma)\delta(\mathrm{R}^n) , \tag{25}$$

where R^n denotes a word with n R's.

The immediate consequence of lemma 1 is that the parity of any even power of a word σ is zero, namely,

$$\delta(\sigma^{*2n}) = 0 , \tag{26}$$

and the parity of any odd power of a word σ depends on $J(\sigma)$ as follows,

$$\begin{aligned} \delta(\sigma^{*2n+1}) &= 1 , \quad \text{if } J(\sigma) \text{ is odd} , \\ &= 0 , \quad \text{if } J(\sigma) \text{ is even} , \end{aligned} \tag{27}$$

i.e.,

$$\delta(\sigma^{*2n+1}) = \tfrac{1}{2}[1 - (-1)^{J(\sigma)}] . \tag{28}$$

Given any two periodic words on the same periodic trajectory, $\sigma(p)$ and $\omega(p)$, suppose $\sigma(p) \succ \omega(p)$, $\sigma(p)$ and $\omega(p)$ are said to be adjacent or consecutive if there exists no other word $\rho(p)$ on the same trajectory such that $\sigma(p) \succ \rho(p) \succ \omega(p)$. A set is said to be an adjacent or consecutive set of order p if it only consists of adjacent periodic words with period p belonging to the same trajectory. The set is denoted as $\Gamma(p)$. Considering two non-intersecting adjacent sets, we defined the ordering of two maximal (or minimal) words of the two sets is the ordering of the two sets, namely, if $\sigma_{\mathrm{M}}(p) \in \Gamma(p)$, $\sigma'_{\mathrm{M}}(p) \in \Gamma'(p)$ and $\sigma_{\mathrm{M}}(p) \succ \sigma'_{\mathrm{M}}(p)$, then

$\Gamma(p) \succ \Gamma'(p)$. Also, two non-intersecting adjacent sets are said to be adjacent to each other if the union of them is still an adjacent set.

Lemma 2. For a given periodic word $\sigma \equiv \sigma(p+1)$ and $k \in \{0, 1, 2, ..., (p+1)^n - 1\}$, the set $\{\varphi^k(\sigma^{*n}) \mid \mathscr{Q}^r_{p+1}(k) = m\}$ is an adjacent set, with fixed n, r and m, where $n \in \{2, 3, ...\}$, $r \in \{1, 2, ..., n-1\}$ and $m \in \{0, 1, ..., (p+1)^r - 1\}$.

The set

$$\Delta^m_{n,r}(\sigma(p+1)) = \{\varphi^k(\sigma^{*n}(p+1)) \mid \mathscr{Q}^r_{p+1}(k) = m\} , \tag{29}$$

with a fixed m and $k \in \{0, 1, ..., (p+1)^n - 1\}$, is called the m-congruence set (on the trajectory $\{\varphi^k(\sigma^{*n}(p+1))\}$) in modular $(p+1)^r$. What lemma 2 states is that an m-congruence set is an adjacent set. The maximum number of adjacent sets on the trajectory $\{\varphi^k(\sigma^{*n}(p+1))\}$ is $(p+1)^{n-1}$, and the minimum number is $(p+1)$. The number of adjacent sets can be varied by adjusting r.

Theorem 1. For a given periodic word $\sigma(p+1)$ and $0 < i, j < p+1$, if

$$\varphi^{i(p+1)^k}(\sigma^{*n}(p+1)) \succ \varphi^{j(p+1)^k}(\sigma^{*n}(p+1)) , \tag{30}$$

for fixed n and k, with $n \in \{2, 3, ...\}$ and $k \in \{0, 1, 2, ..., n-1\}$, then

$$\varphi^{i(p+1)^{k+1}}(\sigma^{*n+1}(p+1)) \succ \varphi^{j(p+1)^{k+1}}(\sigma^{*n+1}(p+1)) . \tag{31}$$

A set $\Pi_n \equiv \{\omega \in \Sigma_2 \mid \omega_1 = \sigma_1, \omega_2 = \sigma_2, ..., \omega_n = \sigma_n\}$ with a specified word $\sigma = \sigma_1\sigma_2...$ is called a cylinder set of order n, centered at σ.

Theorem 2. For a given periodic word $\sigma(p)$ and $0 \leqslant i, j < p$ such that $\varphi^i(\sigma(p)), \varphi^j(\sigma(p)) \in \Pi_n$ with n fixed, and

$$\varphi^i(\sigma(p)) \succ \varphi^j(\sigma(p)) , \tag{32}$$

the ordering between $\varphi^{i+m}(\sigma(p))$ and $\varphi^{j+m}(\sigma(p))$ is determined by

$$[\varphi^{i+m}(\sigma(p)) \succ \varphi^{j+m}(\sigma(p))]^{T(\phi^m \cdot \varphi^i(\sigma(p)))} , \tag{33}$$

where $m < n$.

Theorems 1 and 2 can be thought of as the multiplication and addition rules for ordering of points on a periodic trajectory which is produced by the shift operation. According to the above theorems, we can obtain an explicit expression for the ordering of the points on the trajectory $\{\varphi^k(\sigma^{*n}(p+1))\}$, with arbitrary large period $(p+1)$ and n. Suppose we are given a (superstable) periodic word $\sigma = \sigma(p+1)$, we can establish a mapping $\pi: \mathbb{N} \mapsto \mathbb{N}$, by the MSS order, such that the set

$$\varphi^0(\sigma), \varphi^1(\sigma), \varphi^2(\sigma), ..., \varphi^p(\sigma) \tag{34}$$

is ordered as

$$\varphi^{\pi(0)}(\sigma) \succ \varphi^{\pi(1)}(\sigma) \succ \varphi^{\pi(2)}(\sigma) \succ ... \succ \varphi^{\pi(p)}(\sigma) , \tag{35}$$

where $\pi(0) = 0$ because the word σ is always chosen to be a maximal word σ_M. Also, it can be shown that $\pi(p) = 1$. The set $\{\varphi^k(\sigma^{*n})\}$ represents the trajectory of the nth order bifurcation of $\{\varphi^k(\sigma)\}$. It follows directly from theorems 1 and 2 that the ordering on the nth order bifurcation trajectory $\{\varphi^k(\sigma^{*n})\}$ can be established recursively from the ordering on the original periodic trajectory $\{\varphi^k(\sigma)\}$. Suppose that the mapping $\pi_n: \mathbb{N} \mapsto \mathbb{N}$

(with $\pi_n(0)=0$ and $\pi_n((p+1)^n-1)=1$) orders the trajectory $\{\varphi^k(\sigma^{*n})\}$, with arbitrary n, such that $\varphi^{\pi_n(i)}(\sigma^{*n})\succ\varphi^{\pi_n(i+1)}(\sigma^{*n})$ for $0\leqslant i<(p+1)^n-1$, and the trajectory $\{\varphi^k(\sigma^{*n})\}$ is divided into $p+1$ adjacent sets as the following:

$$
\begin{aligned}
&\Gamma_0((p+1)^n)\equiv\{\varphi^{\pi_n(0)}(\sigma^{*n})\succ\varphi^{\pi_n(1)}(\sigma^{*n})\succ\ldots\succ\varphi^{\pi_n((p+1)^{n-1}-1)}(\sigma^{*n})\}\\
&\quad\succ\Gamma_1((p+1)^n)\equiv\{\varphi^{\pi_n((p+1)^{n-1})}(\sigma^{*n})\succ\varphi^{\pi_n((p+1)^{n-1}+1)}(\sigma^{*n})\succ\ldots\succ\varphi^{\pi_n(2(p+1)^{n-1}-1)}(\sigma^{*n})\}\\
&\quad\succ\ldots\\
&\quad\succ\Gamma_i((p+1)^n)\equiv\{\varphi^{\pi_n(i(p+1)^{n-1})}(\sigma^{*n})\succ\varphi^{\pi_n(i(p+1)^{n-1}+1)}(\sigma^{*n})\succ\ldots\succ\varphi^{\pi_n((i+1)(p+1)^{n-1}-1)}(\sigma^{*n})\}\\
&\quad\succ\ldots\\
&\quad\succ\Gamma_p((p+1)^n)\equiv\{\varphi^{\pi_n(p(p+1)^{n-1})}(\sigma^{*n})\succ\varphi^{\pi_n(p(p+1)^{n-1}+1)}(\sigma^{*n})\succ\ldots\succ\varphi^{\pi_n((p+1)^{n}-1)}(\sigma^{*n})\}\,.
\end{aligned}
\tag{36}
$$

Then the ordering on the trajectory $\{\varphi^k(\sigma^{*n+1})\}$ can be reduced as the following:

$$
\begin{aligned}
&\Gamma'_0((p+1)^{n+1})\equiv\{[\varphi^{\pi_n(0)(p+1)+\pi_n(0)}(\sigma^{*n+1})\succ\varphi^{\pi_n(1)(p+1)+\pi(0)}(\sigma^{*n+1})\succ\ldots\\
&\qquad\succ\varphi^{\pi_n((p+1)^{n-1}-1)(p+1)+\pi_n(0)}(\sigma^{*n+1})]^{T(\bar\varphi^{\pi_n(0)}(\sigma))}\}\\
&\quad\succ\Gamma'_1((p+1)^{n+1})\equiv\{[\varphi^{\pi_n(0)(p+1)+\pi_n(1)}(\sigma^{*n+1})\succ\varphi^{\pi_n(1)(p+1)+\pi_n(1)}(\sigma^{*n+1})\succ\ldots\\
&\qquad\succ\varphi^{\pi_n((p+1)^{n}-1)(p+1)+\pi_n(1)}(\sigma^{*n+1})]^{T(\bar\varphi^{\pi_n(1)}(\sigma))}\}\\
&\quad\succ\ldots\\
&\quad\succ\Gamma'_i((p+1)^{n+1})\equiv\{[\varphi^{\pi_n(0)(p+1)+\pi_n(i)}(\sigma^{*n+1})\succ\varphi^{\pi_n(1)(p+1)+\pi_n(i)}(\sigma^{*n+1})\succ\ldots\\
&\qquad\succ\varphi^{\pi_n((p+1)^{n}-1)(p+1)+\pi_n(i)}(\sigma^{*n+1})]^{T(\bar\varphi^{\pi_n(i)}(\sigma))}\}\\
&\quad\succ\ldots\\
&\quad\succ\Gamma'_p((p+1)^{n+1})\equiv\{[\varphi^{\pi_n(0)(p+1)+\pi_n(p)}(\sigma^{*n+1})\succ\varphi^{\pi_n(1)(p+1)+\pi_n(p)}(\sigma^{*n+1})\succ\ldots\\
&\qquad\succ\varphi^{\pi_n((p+1)^{n}-1)(p+1)+\pi_n(p)}(\sigma^{*n+1})]^{T(\bar\varphi^{\pi_n(p)}(\sigma))}\}\,.
\end{aligned}
\tag{37}
$$

We can illustrate this recurrence rule of ordering the trajectory $\{\varphi^k(\sigma^{*n})\}$ by the example of the first order bifurcation trajectory of a period-5 word $\sigma=$RLRRC. The mapping $\pi_1:\mathbb{N}\mapsto\mathbb{N}$ is: $\pi_1(0)=0$, $\pi_1(1)=3$, $\pi_1(2)=2$, $\pi_1(3)=4$ and $\pi_1(4)=1$. Namely, the trajectory $\{\varphi^k(\sigma=\text{RLRRC})\}$ is ordered as $\{\varphi^0(\sigma)\succ\varphi^3(\sigma)\succ\varphi^2(\sigma)\succ\varphi^4(\sigma)\succ\varphi^1(\sigma)\}$. Then, according to the rule, it is straightforward to write down the ordering on the first order bifurcation trajectory $\{\varphi^k(\sigma^{*2})\}$ explicitly:

$$
\begin{aligned}
&\Omega_0=\{\varphi^0(\sigma^{*2})\succ\varphi^{15}(\sigma^{*2})\succ\varphi^{10}(\sigma^{*2})\succ\varphi^{20}(\sigma^{*2})\succ\varphi^{5}(\sigma^{*2})\}\\
&\quad\succ\Omega_1=\{\varphi^3(\sigma^{*2})\succ\varphi^{18}(\sigma^{*2})\succ\varphi^{13}(\sigma^{*2})\succ\varphi^{23}(\sigma^{*2})\succ\varphi^{8}(\sigma^{*2})\}\\
&\quad\succ\Omega_2=\{\varphi^7(\sigma^{*2})\succ\varphi^{22}(\sigma^{*2})\succ\varphi^{12}(\sigma^{*2})\succ\varphi^{17}(\sigma^{*2})\succ\varphi^{2}(\sigma^{*2})\}\\
&\quad\succ\Omega_3=\{\varphi^9(\sigma^{*2})\succ\varphi^{24}(\sigma^{*2})\succ\varphi^{14}(\sigma^{*2})\succ\varphi^{19}(\sigma^{*2})\succ\varphi^{4}(\sigma^{*2})\}\\
&\quad\succ\Omega_4=\{\varphi^6(\sigma^{*2})\succ\varphi^{21}(\sigma^{*2})\succ\varphi^{11}(\sigma^{*2})\succ\varphi^{16}(\sigma^{*2})\succ\varphi^{1}(\sigma^{*2})\}\,.
\end{aligned}
\tag{38}
$$

Note that because of $\delta(\bar\varphi^{\pi_1(0)}(\sigma))=0$, $\delta(\bar\varphi^{\pi_1(1)}(\sigma))=0$, $\delta(\bar\varphi^{\pi_1(2)}(\sigma))=1$, $\delta(\bar\varphi^{\pi_1(3)}(\sigma))=1$ and $\delta(\bar\varphi^{\pi_1(4)}(\sigma))=1$, therefore the ordering on the set Ω_2, Ω_3 and Ω_4 has been reversed by the operator T. It is worthwhile to point out that the ordering on the adjacent sets on the $(n+1)$th order bifurcation trajectory $\{\varphi^k(\sigma^{*n+1})\}$ has some similarity to the ordering on the nth order bifurcation trajectory $\{\varphi^k(\sigma^{*n})\}$. This reflects the self-similar structure of the bifurcation tree. In fact, we have

Corollary 1. The mapping $\pi_n:\mathbb{N}\mapsto\mathbb{N}$ has the following property:

$$|\pi_n((m+1)(p+1)^r-1)-\pi_n(m(p+1)^r)|=(p+1)^r\,, \tag{39}$$

where $r\in\{1, 2, ..., n-1\}$ and $m\in\{0, 1, ..., (p+1)^{n-r}-1\}$.

Also, because the word σ is always chosen to be a maximum word, it can be shown that $\varphi(\sigma_M)$ is a minimum word. This leads to $\pi(0)=0$ and $\pi(p)=1$. In consequence, immediately a more general result follows: $\pi_n(0)=0$ and $\pi_n((p+1)^n-1)=1$.

According to eq. (37), the mapping π_{n+1} can be expressed in terms of π_n. Therefore, the mapping π_n, with arbitrary large n, can be written in terms of $\pi\equiv\pi_1$ in general by using the recurrence rule. Namely, for a given word $\sigma\equiv\sigma(p+1)$ with arbitrary period $p+1$, the ordering on trajectory $\{\varphi^k(\sigma)\}$ can be easily established; consequently, the ordering on the nth order bifurcation trajectory $\{\varphi^k(\sigma^{*n})\}$ can be expressed in terms of the ordering on the original trajectory $\{\varphi^k(\sigma)\}$. In order to obtain an explicit formula of π_n in terms of $\pi\equiv\pi_1$, we need to introduce a sequence number inverse operator τ:

$$\pi_n^\tau(i)=\pi_n((p+1)^n-1-i)\,. \tag{40}$$

The operator τ can depend on the invariant coordinate $\Theta(l)\equiv\Theta(\bar{\varphi}^l(\sigma))$ as the following:

$$\begin{aligned}\tau(\Theta(l))&=\tau\,,\quad \text{if}\,\Theta(l)=+1\,,\\ &=1\,,\quad \text{if}\,\Theta(l)=-1\,,\end{aligned} \tag{41}$$

where 1 denotes the identity operator. The product of two operators $\tau(\Theta(l_1))$ and $\tau(\Theta(l_2))$ is

$$\begin{aligned}\tau(\Theta(l_1))\tau(\Theta(l_2))&=\tau(\Theta(l_1))\,,\qquad \text{if}\,\Theta(l_2)=+1\,,\\ &=\tau(-\Theta(p-l_1))\,,\quad \text{if}\,\Theta(l_2)=-1\,.\end{aligned} \tag{42}$$

Note that the product is left-handed, i.e.,

$$\tau(\Theta(l_1))\tau(\Theta(l_2))\tau(\Theta(l_3))\equiv\tau(\Theta(l_1))(\tau(\Theta(l_2))\tau(\Theta(l_3)))\,. \tag{43}$$

The idea behind the introduction of the operator τ is that it converts the operation on an ordered sequence of words to the operation on the mapping π_n, which is an operation of numbers. Therefore, the ordering on the sequence becomes a simple calculation of π_n^τ.

Now we can give a general formula of π_{n+1} in terms of π to conclude this section. Given $\sigma=\sigma(p+1)$ and an integer i, $0\leqslant i<(p+1)^{n+1}$ for a fixed n, then

$$\begin{aligned}\pi_{n+1}(i)=&(p+1)^n\pi(l_0)^{\tau(\Theta(l_1))\tau(\Theta(l_2))...\tau(\Theta(l_n))}+(p+1)^{n-1}\pi(l_1)^{\tau(\Theta(l_2))\tau(\Theta(l_3))...\tau(\Theta(l_n))}+...\\ &+(p+1)^{n-k}\pi(l_k)^{\tau(\Theta(l_{k+1}))\tau(\Theta(l_{k+2}))...\tau(\Theta(l_n))}+...+(p+1)\pi(l_{n-1})^{\tau(\Theta(l_n))}+\pi(l_n)\,,\end{aligned} \tag{44}$$

where $l_k=\mathscr{P}^k_{p+1}(i)$, i.e., the integer i is written in mod$(p+1)$ as

$$i=l_0+l_1(p+1)+l_2(p+1)^2+...+l_k(p+1)^k+...+l_n(p+1)^n\,. \tag{45}$$

This explicit formula for the ordering of the points on the nth level bifurcation trajectory will simplify the calculation of the fractal dimension of critical periodic points with arbitrary large period.

The ordering of points on a periodic trajectory of a map is directly related to the topological entropy of the dynamical system described by the map, because the eigenvalues of the order correlation matrix, i.e., the Stefan matrix, determine the topological entropy. Thus, having obtained the ordering of points on periodic trajectories of unimodal mapping, the topological entropy and some associated quantities, which are believed to be important for investigating chaotic behavior of the dynamical systems [6], can be calculated. Indeed, in a previous work [1] by Peng and Cao, the ordering of points on a periodic trajectory of a unimodal map has been applied to accomplish the numerical calculations.

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