ENGINEERING DECISIONS/HYPOTHESIS TESTING

Bayes' Decision Strategy

There are several features we must first stipulate before making a decision:

1. the hypotheses from which to choose,
2. the possible ways that outside influences may enter the problem,
3. a criterion to be used in choosing the best course of action to follow

e.g., suppose we wish to make a decision about the radar signal used to detect an incoming missile. Typically, the electromagnetic wave will be transmitted/reflected through a region with electromagnetic noise (e.g., lightning rf signals bouncing around, reflections from a flock of geese ....).
The basic question is whether the received signal = detection of an incoming missile.

• Possible hypotheses : "signal is present" , or "signal is not present"
• Outside Influence: "noise component"
• costs in making a decision : must be somehow assigned
[e.g., cost = 0 if the correct decision is made, cost = 1 if incorrect decision is made]
• Decision rule: choose "no signal" if the receiver output < threshold,
choose "signal" if the receiver output > threshold.

Consider the decision theory between 2 alternatives, using the criterion of minimum average cost.

hypothesis 0 (H_0)
hypothesis 1 (H_1)

There are 4 costs of making a decision:

c_{00} : cost of deciding in favor of H_0 when H_0 actually true
c_{01} : .... H_0 when H_1 ....
c_{10} : .... H_1 when H_0 ....
c_{11} : .... H_1 when H_1 ....

Suppose H_0 was actually true, the conditional average cost of making a decision, $C(D|H_0)$ is

$$C(D|H_0) = c_{00}\,P[\text{decide } H_0 \mid H_0 \text{ true}] + c_{10}\,P[\text{decide } H_1 \mid H_0 \text{ true}] \qquad (12.1)$$

since we can decide either H_0 or H_1.

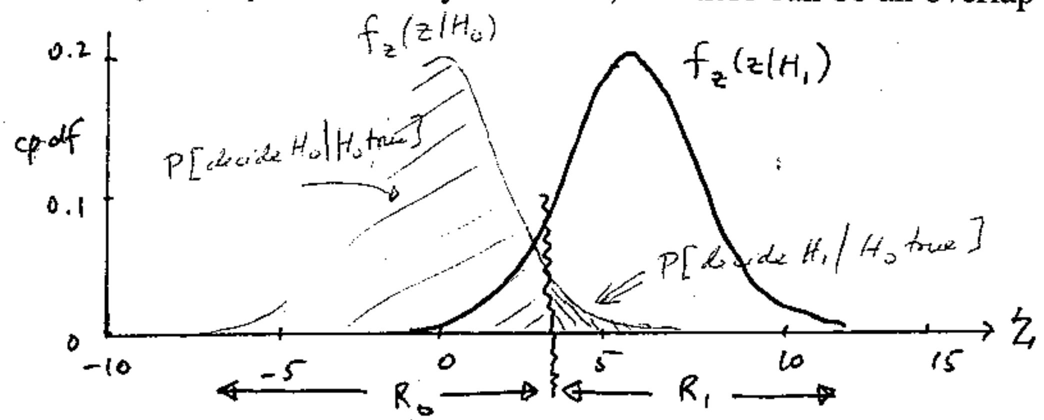
If H_1 was actually true, the conditional average cost of making a decision, $C(D|H_1)$,

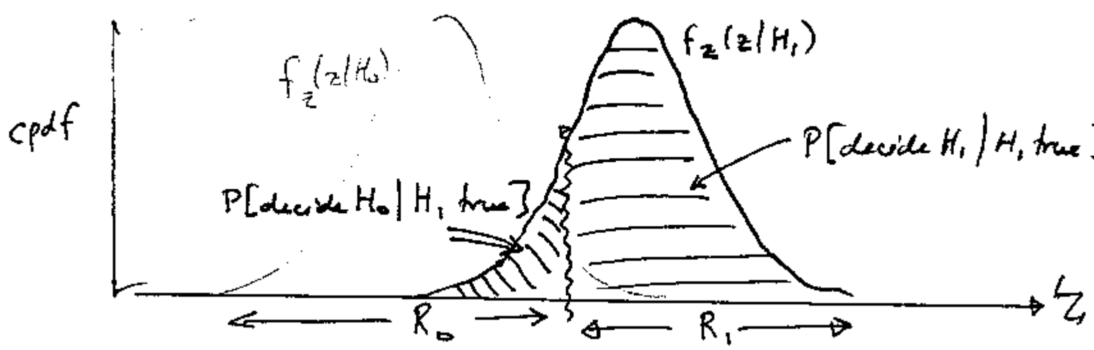
$$C(D|H_1) = c_{01}\,P[\text{decide } H_0 \mid H_1 \text{ true}] + c_{11}\,P[\text{decide } H_1 \mid H_1 \text{ true}] \qquad (12.2)$$

- Suppose the data that is observed is characterized by a R.V Z, with conditional pdf $f_Z(z|H_0)$ if H_0 is true, and cpdf $f_Z(z|H_1)$ if H_1 is true.

- Assume that a decision is made in favor of H_0 if the values for R.V Z falls into some region R_0 of the real line $\mathbb{R}$,
 while a decision for H_1 is made if R.V Z lies in some region R_1

- regions R_0 and R_1 are mutually exclusive, but there can be an overlap of the cpdf's





- Suppose H_0 is true, then the probability of making a correct decision is

$$P[\text{decide } H_0 \mid H_0 \text{ true}] = \int_{R_0} dz\ f_Z(z \mid H_0) \tag{12.3}$$

while the probability of making a wrong decision is

$$P[\text{decide } H_1 \mid H_0 \text{ true}] = \int_{R_1} dz\ f_Z(z \mid H_0) \tag{12.4}$$

Since the values of the R.V Z must lie in region R_0 or region R_1, so one must make a decision - i.e., we have the normalization condition

$$\int_{R_0} dz\ f_Z(z \mid H_0) + \int_{R_1} dz\ f_Z(z \mid H_0) = 1 \tag{12.5}$$

• Now suppose H_1 is true. The probability of making a correct decision is

$$P[\text{decide } H_1 \mid H_1) = \int_{R_1} dz \ f_Z(z \mid H_1) \tag{12.6}$$

while that for making the wrong decision is

$$P[\text{decide } H_0 \mid H_1) = \int_{R_0} dz \ f_Z(z \mid H_1) \tag{12.7}$$

with

$$\int_{R_0} dz \ f_Z(z \mid H_1) + \int_{R_1} dz \ f_Z(z \mid H_1) = 1 \tag{12.8}$$

•• Now to the real problem!

• ASSUME that the hypothesis H_0 happens with probability p
...... H_1 .... $1 - p = q$

• The overall average cost of making a decision is the average of the conditional costs given by (12.1) and (12.2)

$$C(D) = p.\{ c_{00} P[\text{decide } H_0 \mid H_0] + c_{10} P[\text{decide } H_1 \mid H_0] \} + q. \{ c_{01} P[\text{decide } H_0 \mid H_1] + c_{11} P[\text{decide } H_1 \mid H_1] \} \tag{12.9}$$

Using (12.3)-(12.8) :

$$C(D) = p \cdot \left\{ c_{00} \int_{R_0} dz \ f_Z(z|H_0) + c_{10} \left[1 - \int_{R_0} dz \ f_Z(z|H_0) \right] \right\} + q \cdot \left\{ c_{01} \int_{R_0} dz \ f_Z(z|H_1) + c_{11} \left[1 - \int_{R_0} dz \ f_Z(z|H_1) \right] \right\}$$

which simplifies to

$$C(D) = [\, p \, c_{10} + q \, c_{11} \,] + \int_{R_0} dz \ \{ \underbrace{[\, q. (c_{01} - c_{11}) \, . \, f_Z(z|H_1) \,]}_{[\text{ term I}]} - \underbrace{[\, p \, . \, (c_{10} - c_{00}) \, . \, f_Z(z|H_0) \,]}_{[\text{ term II }]} \} \tag{12.10}$$

• We now wish to minimize (12.10).

Note that $p\, c_{10} + q\, c_{11}$ is fixed.

Since it is more costly to make wrong decisions $\Rightarrow$ $c_{01} > c_{11}$
$c_{10} > c_{00}$

Thus term I ≥ 0 , and term II ≥ 0 (since q, p, f_Z are all non-negative)

• One can minimize the integral (and actually make it negative!] by using the following strategy:

•(a) suppose term I < term II : DECISION - R.V Z has values in R_0
This will make the overall integral in (12.10) NEGATIVE.

But what happens if term I > term II ?
We use the normalizations (12.5), and (12.8) to write (12.10) in the form

$$C(D) = [p\, c_{10} + q\, c_{11}] + q(c_{01} - c_{11})\left[1 - \int_{R_1} dz\, f_z(z \mid H_1) \right]$$
$$- p\,(c_{10} - c_{00})\left[1 - \int_{R_1} dz\, f_Z(z \mid H_0) \right]$$

i.e.,

$$C(D) = [p\, c_{00} + q\, c_{01}] - \int_{R_1} dz \{ [q(c_{01} - c_{11}) f_z(z \mid H_1)] -$$
$$- [p\,(c_{10} - c_{00}) f_Z(z \mid H_0)] \}$$

so that if Term I > Term II, then the integral is Positive -- but there is the overall NEGATIVE sign.
i.e., (b) if term I > term II : DECISION - R.V. Z has values in R_1 (•)

This can also be written :

IF $q.\,(c_{01} - c_{11})\,.\,f_Z(z|H_1) > p\,.\,(c_{10} - c_{00})\,.\,f_Z(z|H_0)$, CHOOSE H_1

IF $q.\,(c_{01} - c_{11})\,.\,f_Z(z|H_1) < p\,.\,(c_{10} - c_{00})\,.\,f_Z(z|H_0)$, CHOOSE H_0 (12.11)

for the observed values of Z.

Rewriting, if for an observed value for Z results in
"likelihood ratio" "threshold"

$$\text{IF} \quad \frac{f_Z(z|H_1)}{f_Z(z|H_0)} > \frac{p\,.\,(c_{10} - c_{00})}{q.\,(c_{01} - c_{11})} : \text{CHOOSE } H_1 \tag{12.12}$$

otherwise
"likelihood ratio" "threshold"

$$\text{IF} \quad \frac{f_Z(z|H_1)}{f_Z(z|H_0)} < \frac{p\,.\,(c_{10} - c_{00})}{q.\,(c_{01} - c_{11})} : \text{CHOOSE } H_0 \tag{12.13}$$

This testing procedure is called **Bayes' Decision Making**

EXAMPLE 1

A 10 V signal is sent to a distant receiver. A Guassian noise (zero mean, and variance 4V) is intermixed with this signal.

The final signal is compared with a threshold of 5 V: if the final signal voltage > 5 V, then one decides that the original signal was sent, but if it is < 5 V, then it is decided that no signal was sent (just noise is detected).

The cost of making a decision: '0' if the correct decision was made [i.e., decision of signal when there was a signal, and decision that there was no signal when there was no signal]. The cost of a incorrect decision is '1'.

The probability of sending a signal p= 0.5 (and of not sending a signal = 1 - p)

What is the average cost of making a decision?

Can it be made smaller by using a different threshold?

solution: from (12.9), the average cost of making a decision is:

$$C = \frac{c_{00}}{2} P[\text{ no } | \text{no}] + \frac{c_{01}}{2} P[\text{no } | \text{ sent}] + \frac{c_{10}}{2} P[\text{sent } | \text{ no}] + \frac{c_{11}}{2} P[\text{sent } | \text{ sent}]$$

where P[no | sent] ≡ P [decide no signal | signal sent] , ...

Now we are given $c_{00} = 0 = c_{11}$, $c_{01} = 1 = c_{10}$, so that

$$C = \frac{1}{2} P[\text{no } | \text{ sent}] + \frac{1}{2} P[\text{sent } | \text{ no}] \qquad (12.14)$$

Now the received signal Z = 10 + N,

where the noise R.V has Guassian pdf with zero mean and variance 4 :

$$f_N(n) = \frac{1}{(8\pi)^{1/2}} \exp\left[-\frac{x^2}{8}\right] \qquad (12.15)$$

- The probability of deciding no signal when an actual signal of 10 V has been sent = probability that 10 + N < 5 = probability that N < -5

since we are given that the threshold is T = 5.

Hence, from the noise pdf, we have

$$P[\text{ no } | \text{ sent}] = \int_{-\infty}^{-5} dx \frac{\exp[-x^2/8]}{(8\pi)^{1/2}} = \int_{5}^{\infty} dx' \frac{\exp[-x'^2/8]}{(8\pi)^{1/2}} \quad \text{, with } x' = -x$$

$$= \int_{2.5}^{\infty} du \frac{\exp[-u^2/2]}{(2\pi)^{1/2}} \quad \text{, with } u = x'/2$$

$$= Q(2.5) \qquad (12.16)$$

Now the probability of deciding a signal when actually NO signal was sent
= probability that $0 + N > 5$
= P [sent | no] = Q(2.5) .... [c.f., 2nd integral in (12.16)] (12.17)

Substituting (12.16) and (12.17) into (12.14) gives the average cost

$$C = Q(2.5) = 0.006 \qquad (\bullet\bullet)$$

Now let the threshold for detection T be variable.

i.e., P[no | sent] = probability that $10 + N < T$ = probab. that $N < T - 10$

P[sent | no] = probability that $0 + N > T$

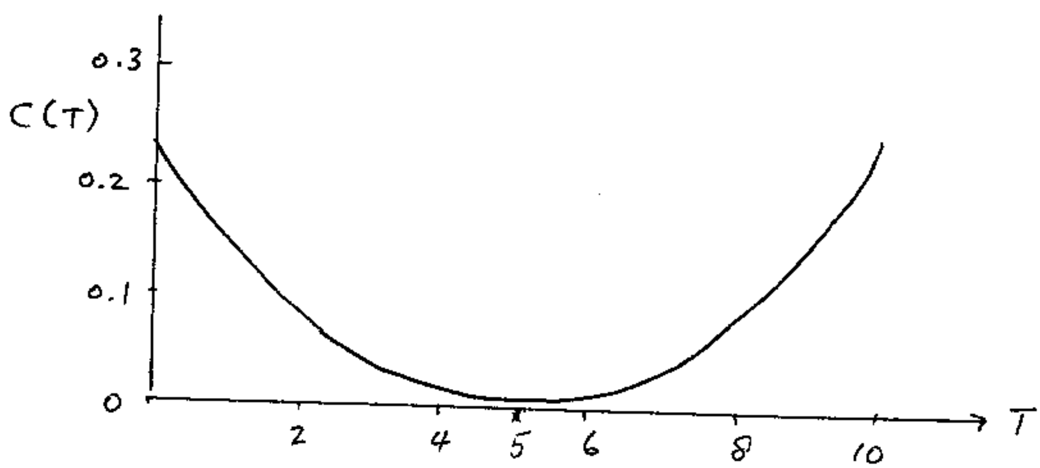
$$P[\text{no} \mid \text{sent}] = \int_{-\infty}^{T-10} dx \frac{\exp[-x^2/8]}{(8\pi)^{1/2}} = \int_{(10-T)/2}^{\infty} du \frac{\exp[-u^2/2]}{(2\pi)^{1/2}} = Q\left(\frac{10-T}{2}\right)$$

$$P[\text{sent} \mid \text{no}] = \int_{T}^{\infty} dx \frac{\exp[-x^2/8]}{(8\pi)^{1/2}} = \int_{T/2}^{\infty} du \frac{\exp[-u^2/2]}{(2\pi)^{1/2}} = Q\left(\frac{T}{2}\right)$$

Hence the average cost as a function of threshold

$$C(T) = \frac{1}{2} Q\left(\frac{10-T}{2}\right) + \frac{1}{2} Q\left(\frac{T}{2}\right) \qquad (12.18)$$

The plot of this function C(T) vs. T is :



Hence with these costs { '0' if correct, '1' if wrong), the minimum average cost occurs at threshold value T = 5V.

Note that the cost is higher if we assume T = 0 threshold voltage!

Example 1 - Generalized

We are trying to detect a signal of arbitrary (but non-random for simplicity!) amplitude A with additive noise N given by a Guassian pdf with mean 0 and variance σ^2.

The costs are the same as before [$c_{00} = c_{11} = 0$; $c_{01} = c_{10} = 1$]. Let the threshold = T.

(a) If an actual signal has been sent, the receiver's signal

$$Z = A + N \tag{12.19}$$

which is also Gaussian, but with mean A and variance σ^2.

Hence the conditional pdf will define, say, the 'hypothesis H_1'

$$\text{hypothesis } H_1 \ : \ \text{cpdf} = f_Z(z \mid H_1) = \frac{\exp[-(z-A)^2/2\sigma^2]}{(2\pi\sigma^2)^{1/2}} \tag{12.20}$$

(b) If no signal is sent, then Z = N

which is Gaussian, but with mean 0 and variance σ^2.

Hence the conditional pdf is

$$\text{hypothesis } H_0 \ : \ \text{cpdf} = f_Z(z \mid H_0) = \frac{\exp[-z^2/2\sigma^2]}{(2\pi\sigma^2)^{1/2}} \tag{12.21}$$

•• **N.B** : it is immaterial what event we call H_0 and H_1. But $p = P[H_0]$, $q = P[H_1]$ ••

The likelihood ratio, see (12.12) or (12.13), is

$$\frac{f_Z(Z \mid H_1)}{f_Z(Z \mid H_0)} = \exp\left[\frac{2AZ - A^2}{2\sigma^2}\right] \tag{12.22}$$

Let p = probability of hypothesis H_0 : i.e., probability that Z = N

q = 1 - p = ...H_1 : i.e., ... Z = A + N

Hence, using Bayes' test, (12.12) or (12.13):

$$\text{IF} \quad \frac{f_Z(z|H_1)}{f_Z(z|H_0)} = \exp\left[\frac{2AZ - A^2}{2\sigma^2}\right] > \frac{p}{q} \Rightarrow \text{hypothesis } H_1 : Z = A + N$$

Taking the natural logarithm of both sides (and remember that the "ln"-fn is monotonic so that it does preserves the inequality direction)

$$H_1 : \frac{2AZ - A^2}{2\sigma^2} > \ln\left[\frac{p}{q}\right] \Rightarrow Z > \frac{\sigma^2}{A}\ln\left[\frac{p}{q}\right] + \frac{A}{2} \tag{12.23}$$

Similarly, Hypothesis H_0 (no signal) if

$$H_0 : \quad Z < \frac{\sigma^2}{A}\ln\left[\frac{p}{q}\right] + \frac{A}{2} \tag{12.24}$$

(a) $p = q = 0.5$

In this case, $\ln [p/q] = 0$, and we get back to the previous example where the best threshold (Z) is A/2, i.e., T = signal amplitude/2

(b) $p \neq q$

As p increases (so that $q = 1 - p$ decreases), $\ln [p/q]$ increases, so that the decision boundary increases and moves to the right ⇒ signal-absent hypothesis should be chosen with greater probability, since the signal-absent case is more probable before an observation of the data is made.

With given costs $c_{00} = c_{11} = 0$; $c_{10} = c_{01} = 1$, the average cost for making a decision

$$C(D) = q \,.\, P[\text{no} \mid \text{sent}] + p \,.\, P[\text{sent} \mid \text{no}]$$

Now the probability of saying that no signal is present when there is

= probability that $Z < (\sigma^2/A) \,.\, \ln (p/q) + A/2 \equiv T$, with pdf (12.20).

-- i.e., Z is below threshold T.

This is the **probability of a miss :**

$$P_{miss} = \int_{-\infty}^{T} dz \frac{\exp\left[-\dfrac{(z-A)^2}{2\sigma^2}\right]}{(2\pi\sigma^2)^{1/2}} = \int_{(A-T)/\sigma}^{\infty} du \frac{\exp(-u^2/2)}{(2\pi)^{1/2}}$$

$$= Q\left(\frac{A-T}{\sigma}\right) \quad , \text{if } T < A$$

$$= 1 - Q\left(\frac{T-A}{\sigma}\right) \quad , \text{if } T > A \qquad (12.25)$$

where we have made the substitution $u = (A - z)/\sigma$.

The probability of saying that the signal was present when there was no signal

= probability that $Z > T$

= **probability of a FALSE ALARM**

$$P_{FA} = \int_{T}^{\infty} dz \frac{\exp\left[-\dfrac{z^2}{2\sigma^2}\right]}{(2\pi\sigma^2)^{1/2}} = \int_{T/\sigma}^{\infty} du \frac{\exp(-u^2/2)}{(2\pi)^{1/2}}$$

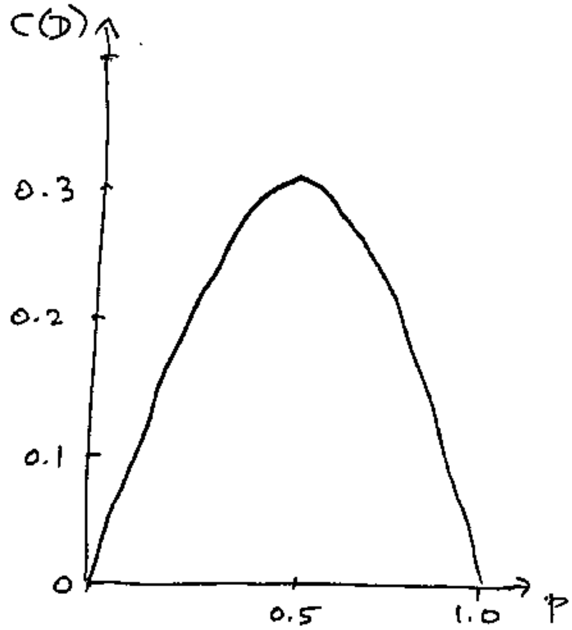
$$= 1 - Q\left(\frac{|T|}{\sigma}\right) \quad , \text{if } T < 0$$

$$= Q\left(\frac{T}{\sigma}\right) \quad , \text{if } T > 0 \qquad (12.26)$$

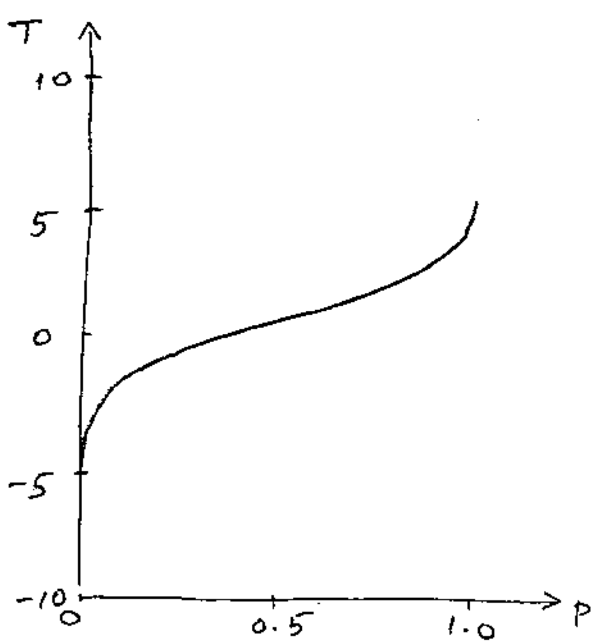
The average cost of a decision = average probability of making an error.

$$C(D) = (1 - p) .\ P_{miss} + p \,.\, P_{FA} \qquad (12.27)$$

With specific values for A and σ (e.g., A = 1, σ = 2), the plot of C(D) as a fn. of p is



The plot of the threshold T as a fn. of p is



When p = 0.5, C(D) is at a maximum, and T = 0.5 = A/2
as shown in the previous example.