The Dirac Equation

2.1 Non-Relativistic QM

Replacing

$$p \to -i\nabla \quad , \quad E \to i\frac{\partial}{\partial t} \qquad (1)$$

Substituting into

$$E = \frac{|p|^2}{2m}$$

we get Schrödinger equation

$$\psi^* \cdot \Big\{ \quad -\frac{1}{2m}\nabla^2\psi = i\frac{\partial}{\partial t}\psi \qquad (2)$$

complex conjugate of Schrödinger equation:

$$\psi \cdot \Big\{ \quad -\frac{1}{2m}\nabla^2\psi^* = -i\frac{\partial}{\partial t}\psi^* \qquad (3)$$

$$-\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left[\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial}{\partial t}\psi^*\right]$$

(4) as: $\psi^*\nabla^2\psi - \psi\nabla^2\psi^* = \nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

then $-\frac{1}{2m}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = i\frac{\partial\psi^*\psi}{\partial t}$

(5) $\nabla\cdot j + \frac{\partial\rho}{\partial t} = 0$ (continuity equation)

(6) $\rho = \psi^*\psi = |\psi|^2$

(7) $j = \frac{1}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

solutions of (2) $\Rightarrow \psi = N e^{i(p\cdot r - Et)}$

then $-i\nabla\psi = p\psi \quad ; \quad i\frac{\partial}{\partial t}\psi = E\psi$

So the plane wave solutions are eigenstates of the momentum and energy operators and hence correspond to free particle.

now if $\psi = N e^{i(pr - Et)} \Rightarrow$ then

$$\rho = |\psi|^2 = N^2$$

$$j = \frac{1}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) =$$

$$= \frac{1}{2im}\left(ip\, N \cdot N e^{i(p \cdot r - Et)} e^{-(\ )} + ip\, N \cdot N e^{-} \cdot e^{+}\right)$$

$$= \frac{p}{m} \cdot |N|^2 = \frac{mv}{m} \cdot |N|^2 = |N|^2 \cdot v$$

from ρ follows that the number of particles per unit volume $= N^2$

from j follows: number of partices per unit time passing through unit area (flux) is $N^2 v$,

hence the current is a vector pointing along direction of motion with magnitude equal to particle flux

The Klein-Gordon Equation

by substitution

$$p = -i\nabla \quad ; \qquad E = -i\frac{\partial}{\partial t}$$

into $\quad p^2 + m^2 = E^2 \qquad (\hbar = c = 1)$

instead of $\frac{p^2}{2m} = E$

we get

$$\left(-\nabla^2 + m^2\right)\psi = -\frac{\partial^2}{\partial t^2}\psi$$

or

$$\boxed{\nabla^2\psi - m^2\psi = \frac{\partial^2}{\partial t^2}\psi} \qquad (8)$$

using: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \nabla\right)$

$$\Box \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$\left(\partial^\mu \partial_\mu + m^2\right)\psi = 0$$

$$\boxed{\left(\Box + m^2\right)\psi = 0} \qquad \text{Klein-Gordon}$$

for plane waves $\psi = N e^{i(p\cdot r - Et)}$
(with p and E constant)

$$\nabla^2\psi = -|p|^2\psi \quad ; \quad \frac{\partial^2\psi}{\partial t^2} = -E^2\psi$$

$$\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = -\left(E^2 - |p^2|\right)\psi$$
$$= -m^2\psi$$

satisfies KG equation

but now $E = \pm\sqrt{p^2+m^2}$

So we have positive and negative energy solutions of KG equation for free particle

both
$$\left.\begin{array}{l} \psi = N \exp[i(p\cdot r - Et)] \\ \psi = N \exp[-i(p\cdot r - Et)] \end{array}\right\} \begin{array}{l} E>0 \\ E<0 \end{array}$$
Satisfy KG equation

the solution

$\left(\frac{\partial}{\partial t} = -E\right)$ are eigenstates of negative energy

Such solutions do not exist in Schrödinger equation

$\left(E \neq -\frac{p^2}{2m}\right)$ non physical

in a 4-vector notation

$$p^\mu = (E, p) \; ; \; x^\mu = (t, r)$$

$$\psi = N \cdot \exp[-i\, p\cdot x] \qquad E>0$$

$$\psi = N \cdot \exp[+i\, p\cdot x] \qquad E<0$$

Further problems in KG equation are related to probability density

Klein - Gordon eq.

$$\nabla^2 \psi - m^2 = \frac{\partial^2}{\partial t^2} \psi \qquad (8)$$

the complex conjugate

$$\nabla^2 \psi^* - m^2 = \frac{\partial^2}{\partial t} \psi^* \qquad (11)$$

$\psi^* \cdot (8) - \psi(11)$ we get

$$\psi^* (\nabla^2 \psi - m^2 \psi) - \psi (\nabla^2 \psi^* - m^2 \psi^*) = \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi}{\partial t}$$

$$\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* - \cancel{m^2 \psi^* \psi} + \cancel{m^2 \psi \psi^*} =$$

$$= \frac{\partial}{\partial t} \left(\psi^* \frac{\partial}{\partial t} - \psi \frac{\partial}{\partial t} \psi^* \right)$$

$$\nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = \frac{\partial}{\partial t} \left(\psi^* \frac{\partial}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right)$$

continuity equation:

$$\nabla \cdot j + \frac{\partial \rho}{\partial t} = 0$$

$$\begin{cases} \rho = i \left(\psi^* \frac{\partial}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \\ \nabla = i (\psi \nabla \psi^* - \psi^* \nabla \psi) \end{cases}$$

for $\psi = N \exp[i (p \cdot r - Et)]$

$\rho = 2 |N|^2 E \quad ; \quad j = 2 |N|^2 p$

ρ (probability density) has the same sign as E

if $E < 0 \Rightarrow \rho < 0$?

The Dirac Equation

look for equation
first order in $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$

$$H\psi = (\alpha \cdot p + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

H is Hamiltonian operator

$p = -i\nabla$ – momentum operator

$$\left[-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z}\right]\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

squaring

$$\left[-i\alpha_x \frac{\partial}{\partial x} - \cdots\right]\left[-i\alpha_x \frac{\partial}{\partial x} - \cdots\right] = \left(i\frac{\partial}{\partial t}\right)\left(i\frac{\partial}{\partial t}\right)\psi$$

we get

$$-\alpha_x^2 \frac{\partial^2}{\partial x^2} - \alpha_y^2 \frac{\partial^2}{\partial y^2} - \alpha_z^2 \frac{\partial^2}{\partial z^2} -$$
$$-(\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2 \psi}{\partial x \partial y}$$
$$-(\alpha_x\alpha_z + \alpha_z\alpha_x)\frac{\partial^2 \psi}{\partial x \partial z}$$
$$-(\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2 \psi}{\partial y \partial z}$$
$$-i(\alpha_x\beta + \beta\alpha_x)m\frac{\partial \psi}{\partial x} - i(\alpha_y\beta + \beta\alpha_y)m\frac{\partial \psi}{\partial y} -$$
$$-i(\alpha_z\beta + \beta\alpha_z)m\frac{\partial \psi}{\partial z} + \beta^2 m^2 \psi = -\frac{\partial^2 \psi}{\partial t^2}$$

but it also must satisfy

Klein - Gordon equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - m^2 \psi = \frac{\partial^2 \psi}{\partial t^2}$$

for this to happen

$$\begin{cases} \beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1 \\ \beta \alpha_j + \alpha_j \beta = 0 \\ \alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k) \end{cases}$$

so α_j and β must be 4×4 matrices

this means that ψ must be a 4-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

(two spin states of $\frac{1}{2}$ particle and two spin states of $\frac{1}{2}$ <u>anti</u>particle)

for H to be Hermitian

α and β also should be Hermitian

$$\alpha_x^\dagger = \alpha_x, \quad \alpha_y^\dagger = \alpha_y, \quad \alpha_z^\dagger = \alpha_z, \quad \beta^\dagger = \beta$$

choice $\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2.4 Probability Density and Current

(23) $$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m \psi = i \frac{\partial \psi}{\partial t}$$

Taking the Hermitian
(remembering that $A^\dagger = (A^*)^T$
so $i^\dagger = -i$ and $(AB)^\dagger = B^\dagger A^\dagger$)

we obtain

(24) $$i \frac{\partial \psi^\dagger}{\partial x} \alpha_x^\dagger + i \frac{\partial \psi^\dagger}{\partial y} \alpha_y^\dagger + i \frac{\partial \psi^\dagger}{\partial z} \alpha_z^\dagger + \psi^\dagger \beta^\dagger m = -i \frac{\partial \psi^\dagger}{\partial t}$$

now $$\psi^\dagger \cdot (23) - (24) \cdot \psi$$

$$\psi^\dagger \left(-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m \psi\right)$$
$$- \left(i \frac{\partial \psi^\dagger}{\partial x} \alpha_x^\dagger + i \frac{\partial \psi^\dagger}{\partial y} \alpha_y^\dagger + i \frac{\partial \psi^\dagger}{\partial z} + \psi^\dagger \beta^\dagger m\right) \psi$$
$$= i \psi^\dagger \frac{\partial \psi}{\partial t} + i \frac{\partial \psi^\dagger}{\partial t} \psi = i \frac{\partial}{\partial t} (\psi^\dagger \psi)$$

Using identity
$$\psi^\dagger \alpha_x \frac{\partial \psi}{\partial x} + \frac{\partial \psi^\dagger}{\partial x} \alpha_x \psi = \frac{\partial}{\partial x} (\psi^\dagger \alpha_x \psi)$$
(and similarly for α_y, α_z)

$$\boxed{\nabla \cdot (\psi^\dagger \alpha \psi) + \frac{\partial}{\partial t} (\psi^\dagger \psi) = 0}$$

$$\begin{cases} \rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0 \\ j = \psi^\dagger \alpha \psi \end{cases}$$

2.5 Spin

Consider the commutator $[H, L]$ where $L = r \times p$ (orbital ang. mom.)

$$[H, L] = [\alpha \cdot p + \beta m, r \times p] =$$

$$= [\alpha \cdot p, r \times p] + \overbrace{\cancel{[\beta m, r \times p]}}^{=0}$$

(since βm is a const. matrix)

$$[H, L] = [\alpha \cdot p, r \times p]$$

$$[H, L_x] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]$$

$$[H, L_y] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, z p_x - x p_z]$$

$$[H, L_z] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, x p_y - y p_x]$$

$$= \cancel{[\alpha_x p_x, y p_z]} - \cancel{[\alpha_x p_x, z p_y]}$$

$$+ \underline{[\alpha_y p_y, y p_z]} - \cancel{[\alpha_y p_y, z p_y]}$$

$$+ \cancel{[\alpha_z p_z, y p_z]} - \underline{[\alpha_z p_z, z p_y]}$$

$$= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] =$$

$$[H, L_x] = -i(\alpha_y p_z - \alpha_z p_y) = -i(\alpha \times p)_x$$

$$[H, L_y] = -i(\alpha \times p)_y, \quad [H, L_z] = -i(\alpha \times p)_z$$

so

$$[H, L] = -i\alpha \times p \qquad (28)$$

L does not commute with $H \Rightarrow$ L is not a constant of motion

Introduce Σ operator:

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

σ – 2×2 Pauli matrices

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \quad \Sigma_y = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

Then $\Rightarrow$

$$[H, \Sigma] = [\alpha\cdot p + \beta m, \Sigma] = [\alpha\cdot p, \Sigma] + [\beta m, \Sigma]$$

but $\Rightarrow$ $[\beta, \Sigma] = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} - \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 0$

then $\Rightarrow$ $[H, \Sigma] = [\alpha\cdot p, \Sigma]$

$$[H, \Sigma_x] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] =$$

$$= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x]$$

Prove:

$$\begin{cases} [\alpha_x, \Sigma_x] = 0 \\ [\alpha_y, \Sigma_x] = -2i\alpha_z \\ [\alpha_z, \Sigma_x] = 2i\alpha_y \end{cases}$$

Then for all components similarly

$$[H, \Sigma_x] = -2i p_y \alpha_z + 2i p_z \alpha_y = 2i(\alpha\times p)_x$$

finally $\left.\begin{array}{l} [H, \Sigma] = 2i\,\alpha\times p \\ \text{from (28)} \Rightarrow [H, L] = -i\,\alpha\times p \end{array}\right\} \Rightarrow [H, L + \tfrac{1}{2}\Sigma] = 0$

$J = L + \frac{1}{2}\Sigma$ is conserved

$\frac{1}{2}\Sigma = S \;\; \| \Rightarrow \;\; J = L + S$

$$S = \frac{1}{2}\Sigma$$

$$S^2 = \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_z = \frac{1}{2}\begin{array}{c} \sigma_z \\ \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{bmatrix} \end{array}$$

$$[S^2, S_z] = 0 \qquad (\underline{\text{Prove}})$$

$$\begin{cases} S(S+1) = \frac{3}{4} & (\text{eigenvalues})\ S^2 \\ S_z = \pm\frac{1}{2} & (\text{eigenvalues of } S_z) \end{cases}$$

(intrinsic electron angular momentum)

2.6 Covariant Notation: the Dirac γ Matrices

$$\gamma^0 \equiv \beta \ ; \ \gamma^1 \equiv \beta \alpha_x \ ; \ \gamma^2 \equiv \beta \alpha_y \ ; \ \gamma^3 \equiv \beta \alpha_z$$

Dirac equation is (eq. 23)

$$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m \psi = i \frac{\partial \psi}{\partial t}$$

multiply by $-\beta$

$$i\beta\alpha_x \frac{\partial \psi}{\partial x} + i\beta\alpha_y \frac{\partial \psi}{\partial y} + i\beta\alpha_z \frac{\partial \psi}{\partial z} - \beta^2 m \psi = -i\beta \frac{\partial \psi}{\partial t}$$

$(\beta^2 = I)$

(31) $$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t}$$

introducing $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

and $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$

$$\gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}$$

then (31) becomes

$$\boxed{\left(i\gamma^\mu \partial_\mu - m\psi\right) = 0}$$

$$\left.\begin{array}{l} (\gamma^0)^2 = 1, \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ \gamma^0\gamma^j + \gamma^j\gamma^0 = 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j = 0 \quad j \neq k \end{array}\right\} \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^{0\dagger} = \gamma^0, \quad \underbrace{\gamma^{1\dagger} = -\gamma^1, \; \gamma^{2\dagger} = -\gamma^2, \; \gamma^{3\dagger} = -\gamma^3}_{\text{anti-Hermitian}}$$

$\downarrow$ Hermitian

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

equations

$$\left\{ \begin{array}{l} \rho = \psi^\dagger \psi \\ j = \psi^\dagger \alpha \psi \end{array} \right\| \Rightarrow \quad j^\mu = \psi^\dagger \gamma^0 \gamma_\mu \psi$$

where $j^\mu = (\rho, j)$

$$\partial_\mu j^\mu = 0$$

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define adjoint spinor

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma^0 = (\psi_1^*, \ldots, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

then the current is:

$$\boxed{j^\mu = \bar{\psi} \gamma^\mu \psi}$$

it can be shown that j^μ transforms as a 4-vector

2.7 Plane Wave Salutions

2.7.1 Derivation

$$\psi = u(E,p)\, e^{i(p \cdot r - Et)}$$

$$\partial_0 \psi = \frac{\partial \psi}{\partial t} = -iE\psi$$

$$\partial_1 \psi = \frac{\partial \psi}{\partial x} = iP_x \psi$$

$$\partial_2 \psi = \frac{\partial \psi}{\partial y} = iP_y \psi$$

$$\partial_3 \psi = \frac{\partial \psi}{\partial z} = iP_z \psi$$

from Dirac equation:

$$\boxed{i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial x} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t}}$$

$$i\gamma^1 i P_x u + i\gamma^2 i P_y u + i\gamma^3 i P_z u - mu = -i\gamma^0 \cdot (-iEu)$$

$$(\gamma^0 E - \gamma^1 P_x - \gamma^2 P_y - \gamma^3 P_z - m)u = 0$$

or

$$(\gamma^\mu P_\mu - m)u = 0$$

$$\gamma^\mu P_\mu - m = \gamma^0 E - \gamma \cdot P - m =$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} E - \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \cdot P - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} =$$

$$\begin{pmatrix} E - m & -\vec{\sigma} \cdot p \\ \vec{\sigma} \cdot p & -E - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(E - m)u_A = (\vec{\sigma} \cdot p)u_B$$

$$(\vec{\sigma} \cdot p)u_A = (E + m)u_B$$