Klein-Gordon Equation

- Apply $\mathbf{p} \rightarrow -i\nabla$, $E \rightarrow i\frac{\partial}{\partial t}$ to $E^2 = p^2 + m^2$:

$$(\partial_\mu \partial^\mu + m^2)\psi = 0$$

Klein-Gordon equation

(Handout 2.2)

- Manifestly Lorentz invariant, **BUT**
  - there are negative energy solutions which can not simply be rejected
  - the probability density can be negative

- These problems led Dirac (1928) to search for an alternative relativistic wave equation

  → Dirac equation

  Spin 1/2 particles and antiparticles

- Problems with K-G equation can be resolved in QFT

  → Spin 0 particles and antiparticles

The Dirac Equation

- Dirac proposed an equation first order in space and time derivatives :

$$\left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi = i\frac{\partial \psi}{\partial t}$$

(Handout 2.3)

- The coefficients $(\alpha_x, \alpha_y, \alpha_z, \beta)$ satisfy

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$

and anticommute :

$$\left.\begin{aligned} \alpha_x\beta + \beta\alpha_x &= 0 \\ \alpha_x\alpha_y + \alpha_y\alpha_x &= 0 \end{aligned}\right\} \quad \text{etc.}$$

$\Longrightarrow$ they are not ordinary numbers, but 4x4 matrices

- ψ is then a 4-component spinor : $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

- The probability density is always +ve :

$$\rho = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$$

(Handout 2.4)

- The Dirac equation describes particles with intrinsic (spin) angular momentum $\hbar / 2$

(Handout 2.5)

- Covariant notation :

(Handout 2.6)

$$\gamma^\mu = \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3\right)$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

$$\gamma^{0+} = \gamma^0 \qquad \gamma^{1+} = -\gamma^1 \qquad \gamma^{2+} = -\gamma^2 \qquad \gamma^{3+} = -\gamma^3$$

- The Dirac equation becomes

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0}$$

- Define the adjoint spinor as

$$\overline{\psi}\equiv\psi^{+}\gamma^{0}=\left(\psi_{1}^{*},\psi_{2}^{*},-\psi_{3}^{*},-\psi_{4}^{*}\right)$$

in terms of which the current is

$$j^{\mu}=\overline{\psi}\gamma^{\mu}\psi$$

- Free particle solutions (Handout 2.7)

Two positive-energy solutions :

$$\psi=\mathrm{u}\left(E,\mathbf{p}\right)e^{i\left(\mathbf{p.r}-Et\right)}$$

Two negative-energy solutions :

$$\psi=\mathrm{v}\left(E,\mathbf{p}\right)e^{-i\left(\mathbf{p.r}-Et\right)}$$

where $E=+\sqrt{|p|^{2}+m^{2}}$ is always > 0

$$\mathrm{v}\left(E,\mathbf{p}\right)=\mathrm{u}(-E,-\mathbf{p})$$

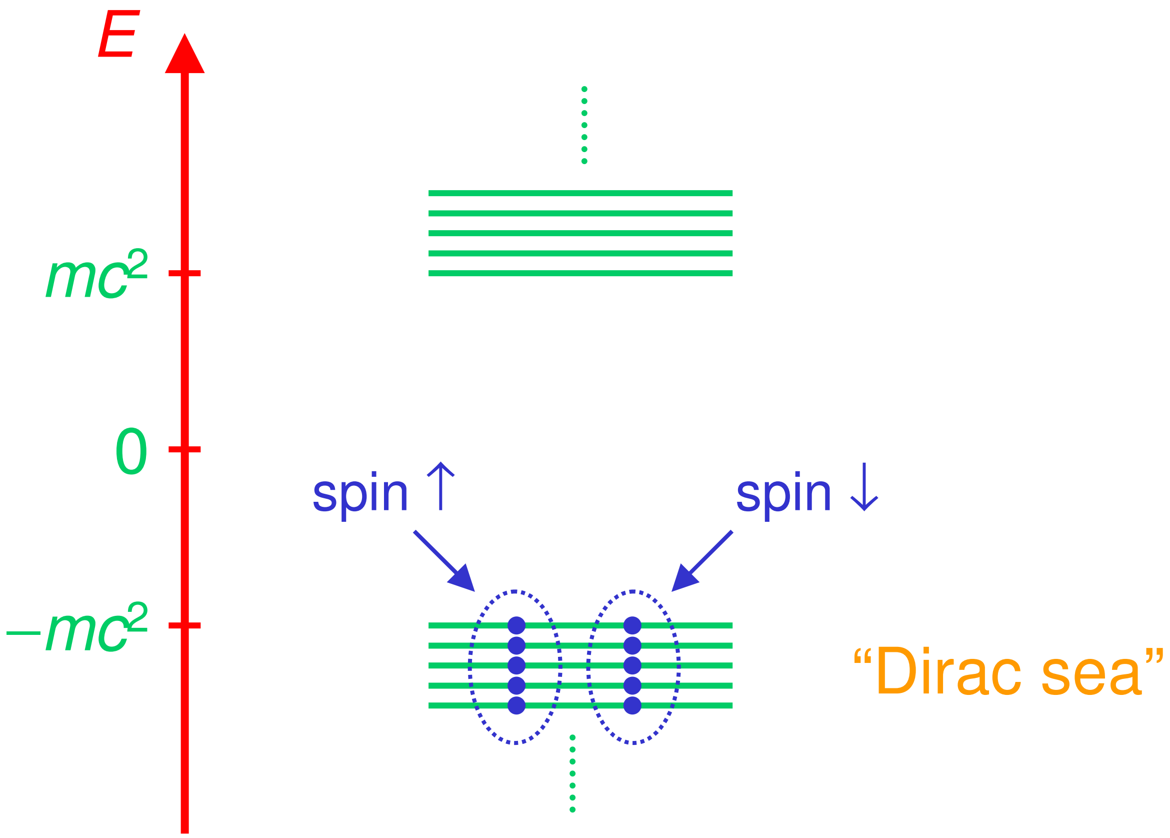
$\Rightarrow$ problem of –ve energy states remains

The Dirac Sea

- Dirac equation has a probability density which is <u>always</u> positive

  BUT: still have negative energy solutions

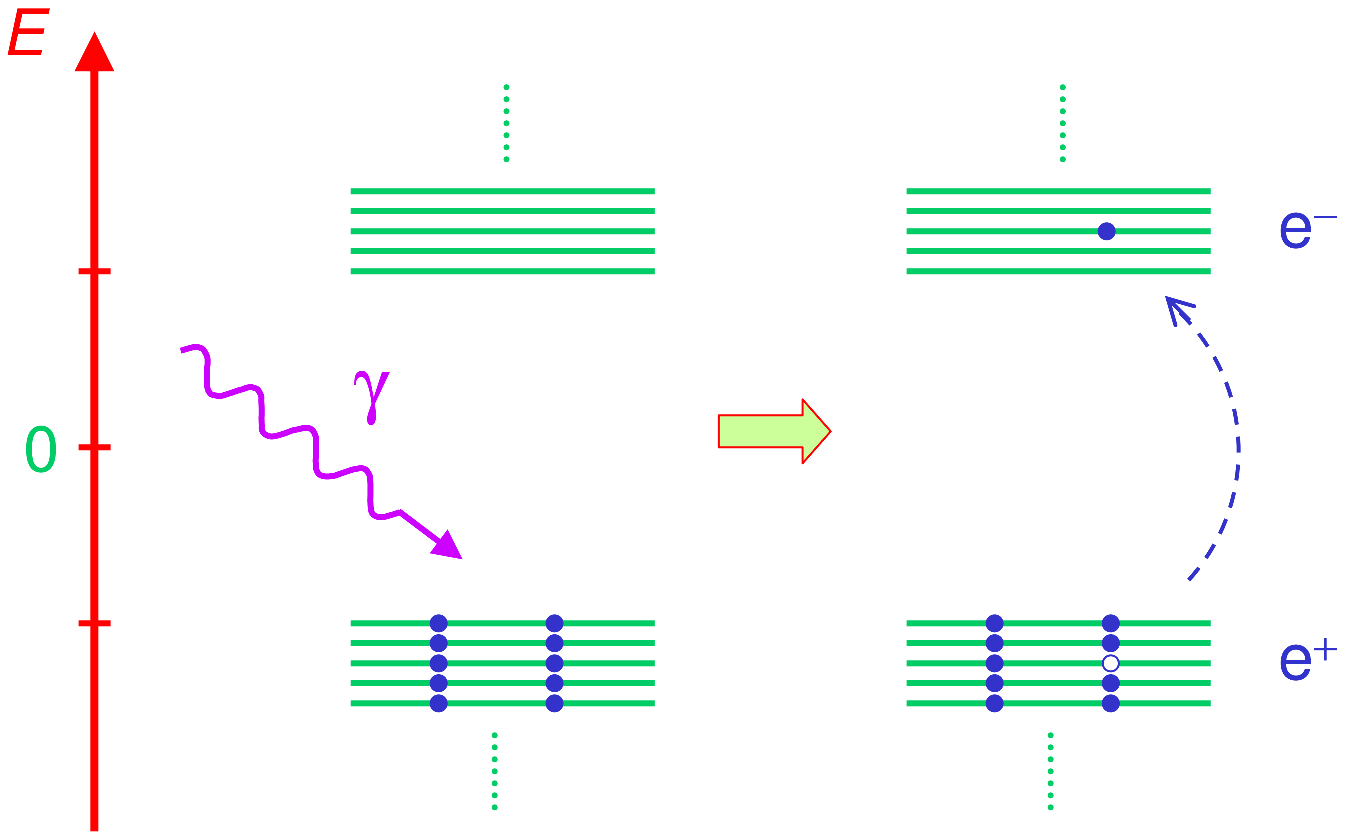
- Dirac: vacuum corresponds to all states with $E < 0$ being occupied by two electrons with opposite spins :



Pauli exclusion principle

$\Longrightarrow$ electrons with $E > 0$ are forbidden from falling into the $E < 0$ states

- A photon could excite an electron from $E < 0$ to $E > 0$:



$\Rightarrow$ creates a <u>hole</u> in the $E < 0$ states:

energy of hole = $-\,E_{hole} > 0$

charge of hole = $-\,Q_e > 0$

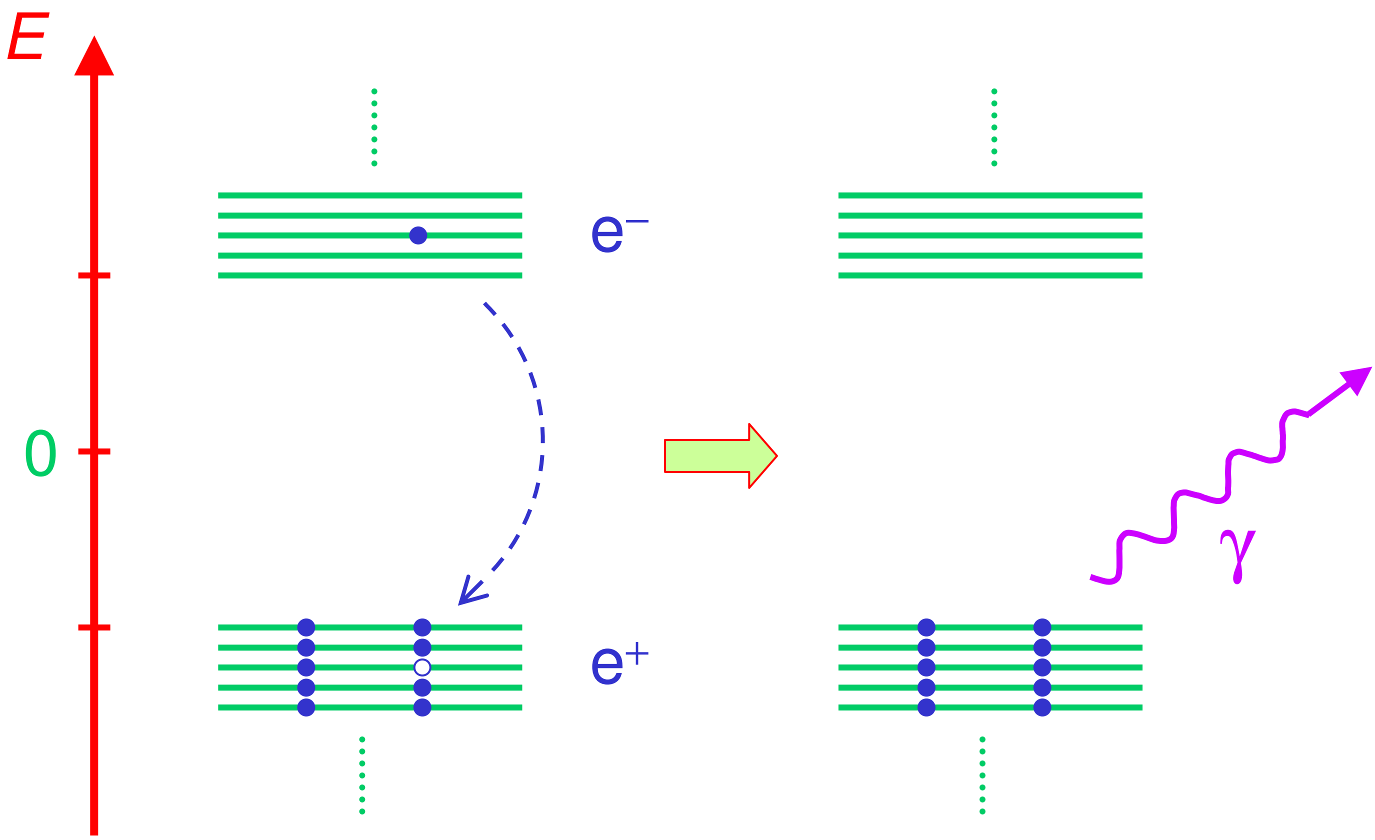
$\Rightarrow$ looks like a positive energy, positively charged electron → "positron" e^+

$\Rightarrow$ Prediction of **antiparticles**

$\gamma \rightarrow e^+ e^-$ pair creation

- Reverse process could also occur :

  electron with $E > 0$ could drop into a hole, emitting a photon:



$e^+ e^- \rightarrow \gamma$ <u>pair annihilation</u>

- But : Dirac sea doesn't work for bosons

  (since exclusion principle doesn't apply)

  $\Longrightarrow$ can't account for spin 0,1 antiparticles

- Schwinger (Weinberg) :

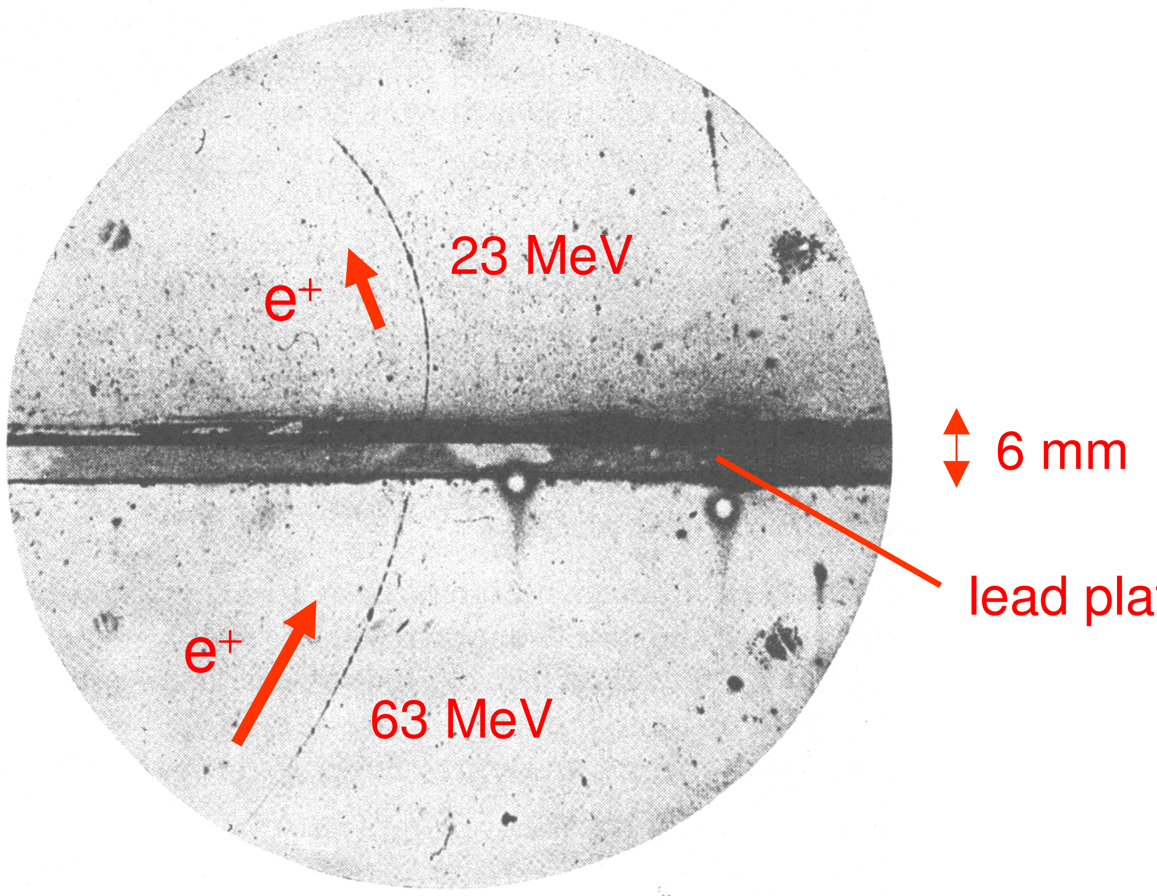
  "The picture of an infinite sea of negative energy electrons is now best regarded as a historical curiosity and forgotten"

Discovery of Positron

C.D.Anderson, Phys Rev **43** (1933) 491

Cosmic ray track in cloud chamber:

(trail of water droplets)

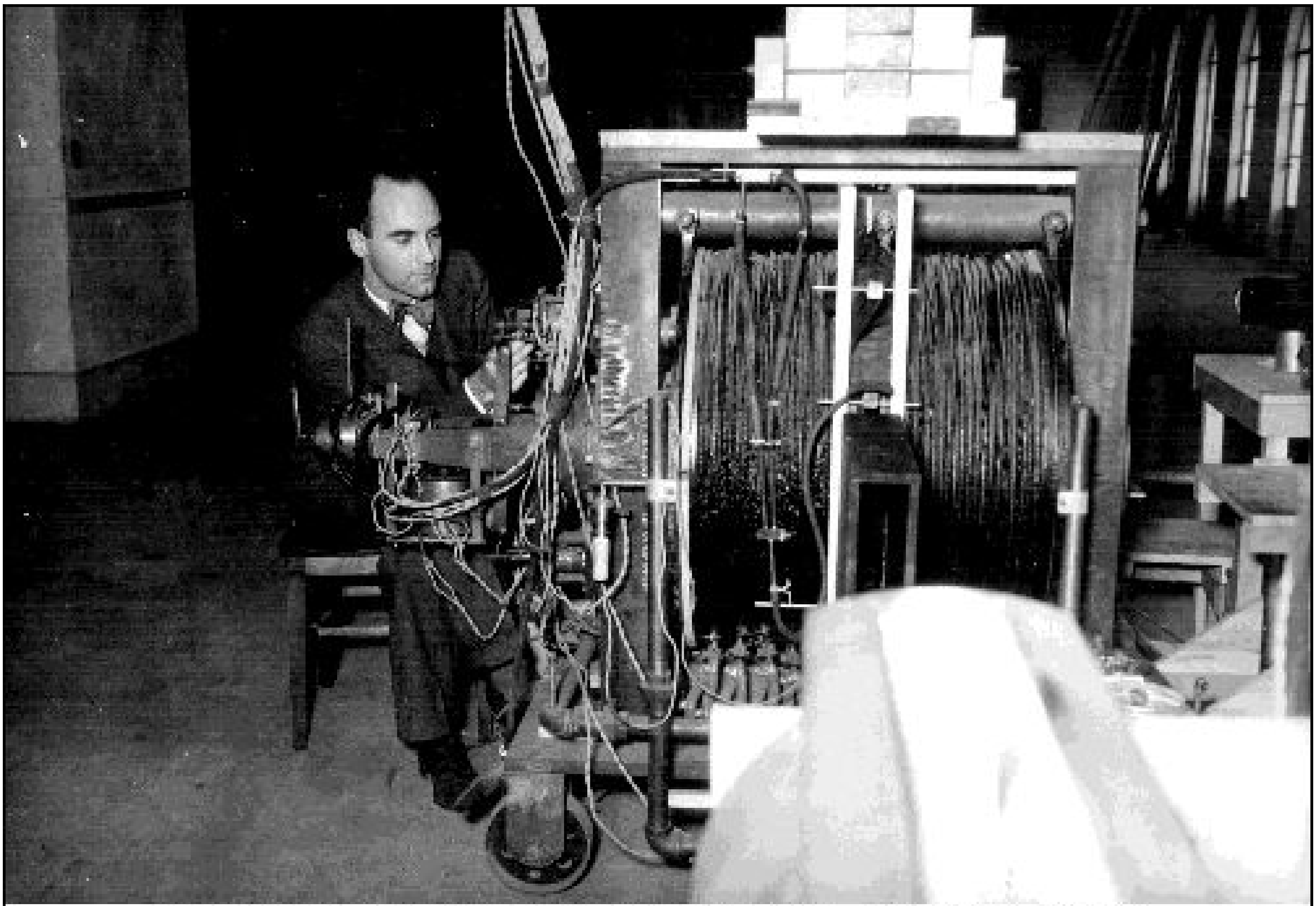


- e^+ enters at bottom, slows down in lead plate
- curvature in ***B*** field gives sign of charge
- long range of upper track $\Longrightarrow$ not a proton

(a proton would only travel ~5 mm)

→ remarkable verification of Dirac's picture

- Anderson’s cloud chamber :



“… seemed to be interpretable only on the basis of the existence of a particle carrying a positive charge but having a mass of the same order of magnitude as that normally possessed by a free negative electron. Later study of the photograph by a whole group of men of the Norman Bridge Laboratory only tended to strengthen this view. …”

Antiparticles

- Feynman interpretation of $E < 0$ solutions :

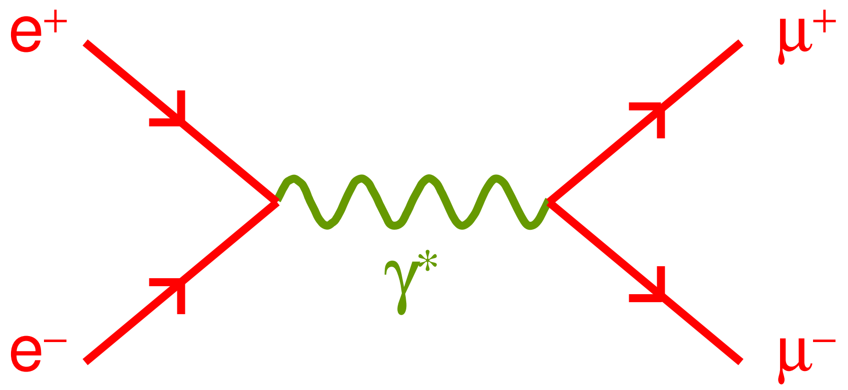
a negative energy particle with

energy $-E < 0$, momentum $-\mathbf{p}$, charge $-q$

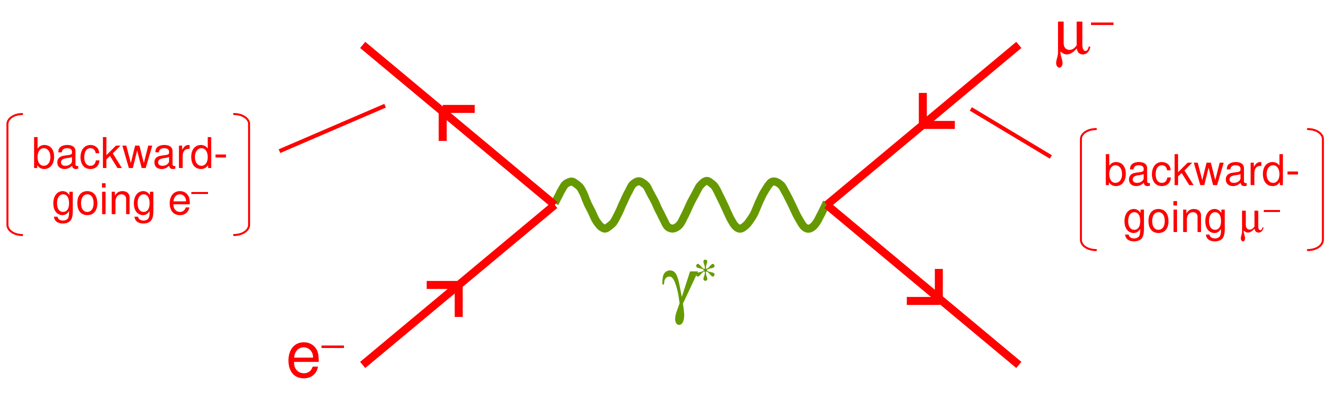
is equivalent to a +ve energy antiparticle with

energy $E > 0$, momentum $\mathbf{p}$, charge q

- For example: $e^+ e^- \rightarrow \mu^+ \mu^-$



can be regarded as:



- Works for bosons also :

the Klein-Gordon equation describes spin 0 particles and antiparticles

Free Particle Solutions

(Handout 2.7)

- Two for particles:

$$\psi = u_i(E,\mathbf{p})e^{i(\mathbf{p}.\mathbf{r}-Et)}$$

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m) \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$

- Two for antiparticles:

$$\psi = \mathrm{v}_i(E,\mathbf{p})e^{-i(\mathbf{p}.\mathbf{r}-Et)}$$

$$\mathrm{v}_1 = N\begin{pmatrix} (p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix} \qquad \mathrm{v}_2 = N\begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}$$

Normalisation: $N = \sqrt{E+m}$

E and **p** are the physical energy and momentum of the particle or antiparticle

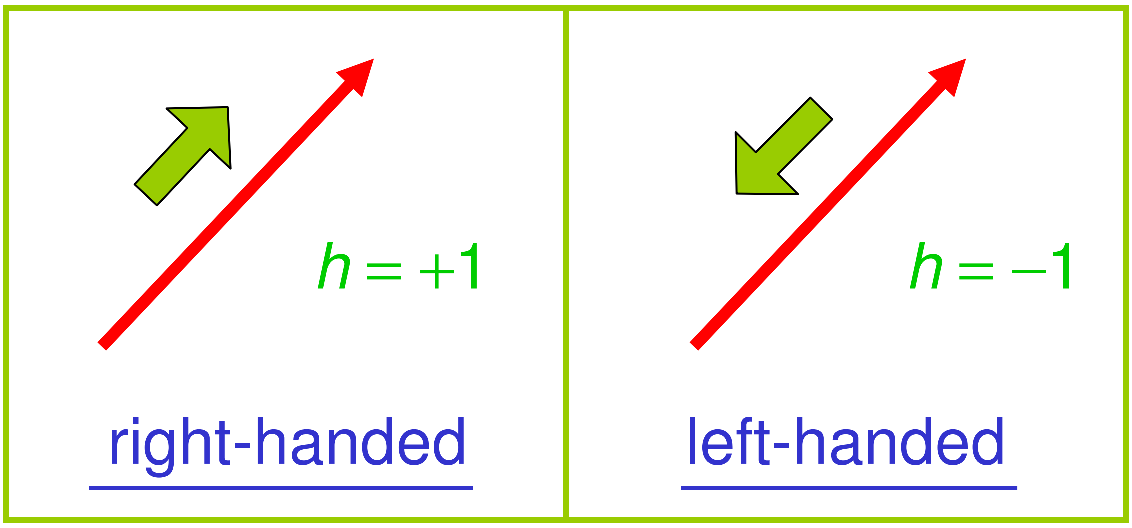
$$E = +\sqrt{|p|^2 + m^2}$$

(i.e. $E > 0$ always)

Helicity

(Handout 2.9)

- Helicity eigenstates :

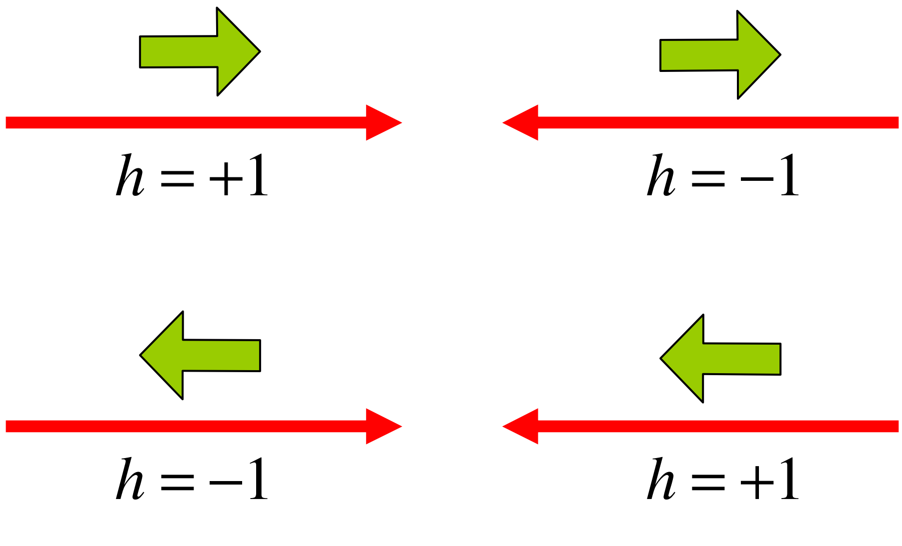


- For a particle or antiparticle (of any energy) travelling along the z axis :

  u_1, v_1, u_2, v_2 are helicity eigenstates :

u_1, v_1: $S_z = +\frac{1}{2}$

u_2, v_2: $S_z = -\frac{1}{2}$



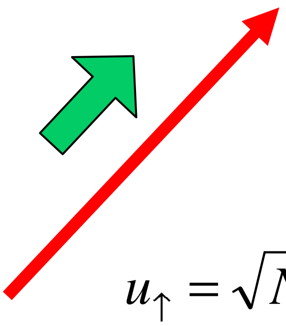
But in any other direction:

u_1, v_1, u_2, v_2 are **not** helicity eigenstates

For motion at an angle θ to the z axis :

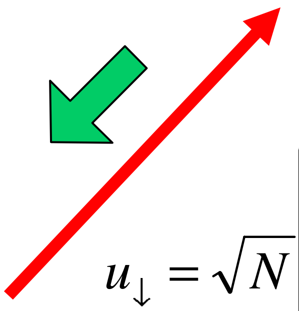
(Handout 2.10)

For a particle :



$$u_\uparrow = \sqrt{N}\begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \\ A\cos\theta/2 \\ Ae^{i\phi}\sin\theta/2 \end{pmatrix}$$

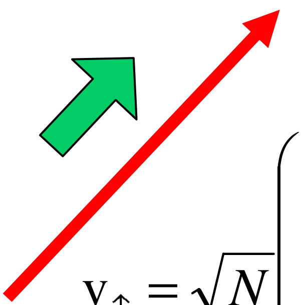
$$h = +1 \quad (\mathrm{RH})$$



$$u_\downarrow = \sqrt{N}\begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \\ A\sin\theta/2 \\ -Ae^{i\phi}\cos\theta/2 \end{pmatrix}$$

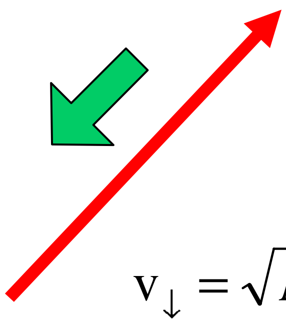
$$h = -1 \quad (\mathrm{LH})$$

For an antiparticle :



$$\mathrm{v}_\uparrow = \sqrt{N}\begin{pmatrix} A\sin\theta/2 \\ -Ae^{i\phi}\cos\theta/2 \\ -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \end{pmatrix}$$

$$h = +1 \quad (\mathrm{RH})$$



$$\mathrm{v}_\downarrow = \sqrt{N}\begin{pmatrix} A\cos\theta/2 \\ Ae^{i\phi}\sin\theta/2 \\ \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$$

$$h = -1 \quad (\mathrm{LH})$$

$$N = E + m$$

$$A \equiv p/(E+m)$$

Chirality

(Handout 2.12)

- The left-handed and right-handed chiral components of a particle or antiparticle spinor ψ are defined as

$$\psi_L = P_L\psi \equiv \tfrac{1}{2}(1-\gamma^5)\psi$$
$$\psi_R = P_R\psi \equiv \tfrac{1}{2}(1+\gamma^5)\psi$$

$$\psi = \psi_L + \psi_R$$

where $$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- In the limit $E \gg m$:

the chiral components ψ_L, ψ_R become helicity eigenstates

For a particle spinor u :

$$u_L = P_L u = \tfrac{1}{2}(1-\gamma^5)u$$ has $h = -1$ (LH)

$$u_R = P_R u = \tfrac{1}{2}(1+\gamma^5)u$$ has $h = +1$ (RH)

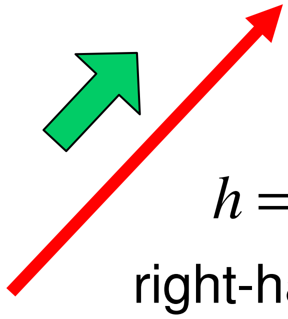
For an antiparticle spinor v :

$$v_L = P_L v = \tfrac{1}{2}(1-\gamma^5)v$$ has $h = +1$ (RH)

$$v_R = P_R v = \tfrac{1}{2}(1+\gamma^5)v$$ has $h = -1$ (LH)

For a particle, with $E \gg m$:

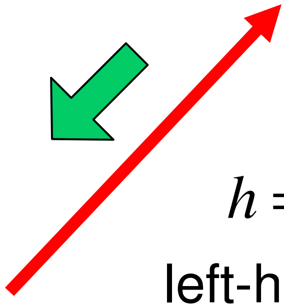
$$P_R u = \tfrac{1}{2}(1+\gamma^5)u$$



$h = +1$

right-handed

$$P_L u = \tfrac{1}{2}(1-\gamma^5)u$$

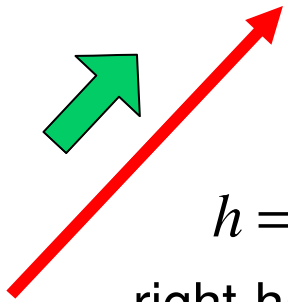


$h = -1$

left-handed

For an antiparticle, with $E \gg m$:

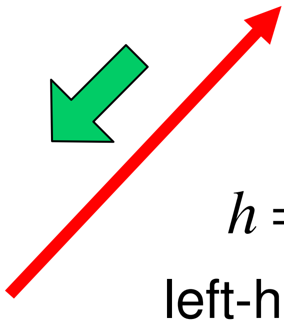
$$P_L \mathrm{v} = \tfrac{1}{2}(1-\gamma^5)\mathrm{v}$$



$h = +1$

right-handed

$$P_R \mathrm{v} = \tfrac{1}{2}(1+\gamma^5)\mathrm{v}$$



$h = -1$

left-handed

Thus, in the relativistic limit, the different uses of the terms "left-handed" and "right-handed" become equivalent

(but note that, for antiparticles :

$P_L \rightarrow$ right-handed helicity eigenstate

$P_R \rightarrow$ left-handed helicity eigenstate)

Dirac Magnetic Moment

(Handout 2.15)

- The Dirac equation predicts that spin half particles have an

  intrinsic (spin) magnetic moment

$$\boldsymbol{\mu} = g\frac{q}{2m}\mathbf{S}$$

Bohr magneton:

$$\mu_B = e/2m$$

where the gyromagnetic ratio $g = 2$

- For electrons :

expt : $g_e/2 = 1.00115965218\,69(41)$

QED : $g_e/2 = 1.00115965213(3)$

- For muons :

expt : $g_\mu/2 = 1.0011659208(6)$

QED : $g_\mu/2 = 1.0011659183(7)$

?

- For protons and neutrons, measure:

$$g_p/2 = +2.79284739(6)$$
$$g_n/2 = -1.9130428(5)$$

$\Rightarrow$ p and n are not pointlike particles