Sparse Numerical Matrix Assembly Algorithms

Input: $\overline{IA}$, $\overline{JA}$ Descriptions of non-zero, off-diagonal terms' locations of matrix [A]

$$\overline{IDIR} = \begin{cases} 1, \text{ corresponds to Dirichlet b.c. location(s)} \\ 0, \text{ elsewhere} \end{cases}$$

$\overline{AE}$, $\overline{BE}$ = Element stiffness matrix & element load vector (if necessary)

$\overline{LM}$ = Global dof numbers, associated with an element

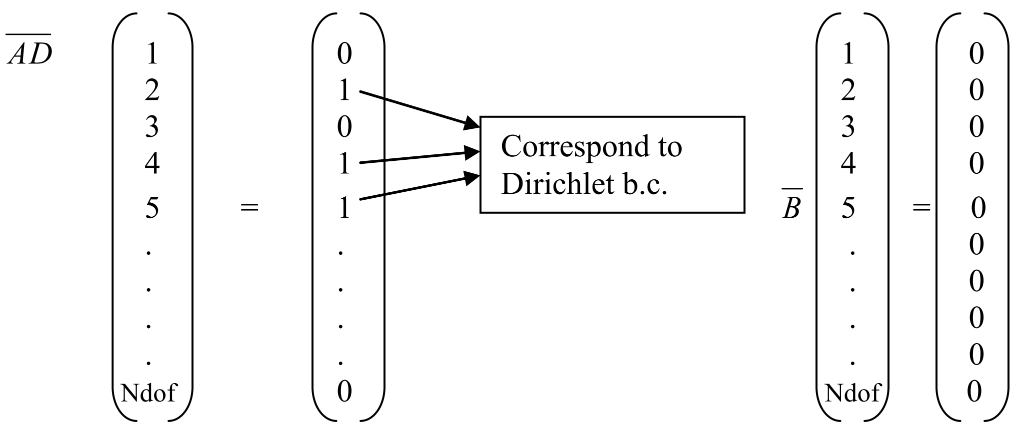
NN = Size of Element Matrix

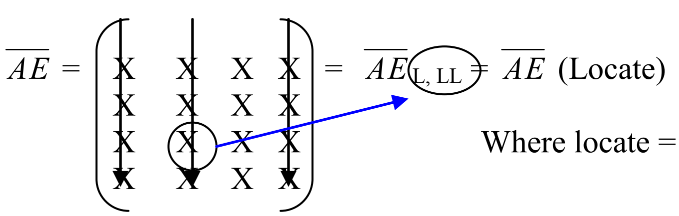
Output: $\overline{AN}$, $\overline{AD}$ = Values of off-diagonal & diagonal terms of [A], respectively

$\overline{B}$ = RHS load vector

Working Space: $\overline{IP}$ (Ndof), should be initialized to $\overline{0}$ before calling this routine (for more efficient)

Notes: $\overline{AD}$ (Ndof) should be initialized to $\overline{0}$ before calling this routine





Where locate = L + (LL-1)*NN

Example: $AE_{3,2}$ is stored in AE (3+1*4)= AE(7)

Key Ideas (For Sparse Numerical Assembly)

Note: Before calling this routine, initialize $\overline{IP}\begin{pmatrix} 1 \\ 2 \\ 3 \\ . \\ Ndof \end{pmatrix} = \bar{0}$

Dϕ 40 L = 1, NN (= Element local row dof #)

- Get I = global row dof # (correspond to L) = LM(L)
- If (I is Dirichlet b.c.) Gϕ Tϕ 40 (⇒ skip)
- Assemble diagonal terms {AD} of [A] from [AE]
- Assemble the I^{th} row of {B}, if we have load acting on element
- KK = 0 ⇒ Indicator that entire row I of [AE] has no contribution to [A]
  For example, entire row I of [AE] is in lower triangle of [A]

Dϕ 20 LL = 1, NN (= element local column dof #)

- Find the location K of $[AE]_{L,LL}$ in the 1-D array of [AE]
- If (LL=L) Gϕ Tϕ 20 ⇒ skip, since diagonal term of [AE] already taken care before loop20
- Get J = LM(LL) = global column dof #
- If (J is Dirichlet b.c.) Gϕ Tϕ 10 ⇒ skip, but modify the RHS vector B first
- If (J.LT.I) Gϕ Tϕ 20⇒ skip lower triangle
- IP(J) = K ⇒ record col.# J (associate with row #I) & the correspond K^{th} location of [AE] = $[AE]_{L,LL}$
- KK = 1 ⇒ Indicator, that row L of [AE] does have contribution to row I of [A]
  Gϕ Tϕ 20

10 B(I) = B(I) – B(J)*AE(K) ⇒ modify RHS B, for Dirich b.c. only

20 Continue

If (KK=0) Gϕ Tϕ 40 ⇒ if row L of [AE] has no contribution

Dϕ 30 J = IA(I) ⇒ IA(I+1)-1 ⇒ pointer to col.# of row of [A]

Icol.# = JA(J)

K =IP(Icol.#)

If (K= 0) Gϕ Tϕ 30 ⇒ skip (the current element has nothing to do with Icol# of [A])

AN(J) = AN(J) + AE(K) ⇒ [A]= Σ[AE]

IP(Icol.#) = 0⇒ initialize again, before considering next row

30 Continue

40 Continue

Important Notes on "Sparse Numerical Assembly"

The elements (terms) of [AE] can't be directly assembled into [A],

For example: right after line 15, because $\overrightarrow{LM}$ and $\overrightarrow{JA}$ (= colume#) are not order!!!

```
01. Dϕ 40 L= 1, NN
02. I = LM(L)
.
.
.
13. If (J.LT.I) Gϕ Tϕ 20
14. IP(J) =K
15. KK = 1
15.1 AN(J) = AN(J) + AE(K)
16.Gϕ Tϕ 20
17. 10 B(I) = B(I) – B(J)*AE(K)
18. 20 continue
.
.
22. Dϕ 30 J = IA(I) , IA(I+1) –1
.
.
25. AN(J) = AN(J) + AE(K)
```

Here, J ≡col.# ⇒ wrong meaning!!! (line 15.1)

Cannot be done here, should be done later on line 25

Here, J ≡ pointer to get column# ⇒ correct meaning (line 25)

October 3, 1999 (from Pierrot)

SERIAL IMPLEMENTATION

- Discussion of the 15k DOF problem:
This problem has 1995 zeros on the diagonal.

The subroutine that implements the new formulation after the numerical factorization is called sover2.f
Time Numfal : 54.45 sec
Time solver02 : 1572.96 sec
Total time solver : 1629.03

Solver02 takes 96.6% of the total sover time
The brake down of time in solver2 (1572.96 sec) is as follows:
- one FBE = 0.774
- Multiple FBE = 1517.38
- [I-Dtrans P] inverse Dtrans X = 0.49
- ROW 8 = 53.12
- X= X* + [P]DI = 1.20

The Relative error is $\| AX - b \| / \| b \|$ = 2.210428581724D-10

MPI Parallel Implementation

Another version of the solver was written implementing to solver in parallel using MPI to mainly solve the multiple RHS in parallel.

The implementation

- All processors receive from the master the factorized matrix
- The column numbers of P are divided equally among the processors. (processor 0 will take also the leftover columns)
- Each processor performs the FBE on its columns
- Instead of using the blocking or non-blocking point to point communication (MPI_SEND/RECV or MPI_ISEND?MPI_IRECV), a collective communication routine (MPI_GATHER) is used instead to collect all the results from processor into matrix P before solving the dense matrix. Processor 0 is the one who is collecting.

Result

The following results are for the15k problem without using MMD and running numfal

	Multiple FBE	SOLVER02	SOLVER00
1 PROC	1556.84	1612.47	1668.99
2 PROCS	795.24	851.75	908.67
4 PROCS	414.51	480.86	537.94
SERIAL	1556.84	1572.97	1629.03

On ≈ Norfolk
Computer@
Minnesota
Supare.inst.sun

The column of Multiple FBE gives the portion that was parallelized and the last one the impact one the overall time.

	Speed up Multiple FBE
1 PROC	1
2 PROCS	1.96
4 PROCS	3.76

TASKS remaining:

1. Put MMD and numfa8 and run all the timing again
2. Use a math library either such lapack and a parallel one from scalapack for solving the dense portion instead of ROWS. This may not be portable though.

After introducing MMD and using numfa8 the times should go down to 5.48 sec for the factorization , solver02 to 538 sec and the total time to 544 sec on a serial run. The multiple FBE taking 464 sec which will then go down using the parallel version. The time for the dense solve is not going down.

Notes:

Multiple FBE (≡ Multiple RHS) + $[I-D^T P]$ + row8 = solver#2

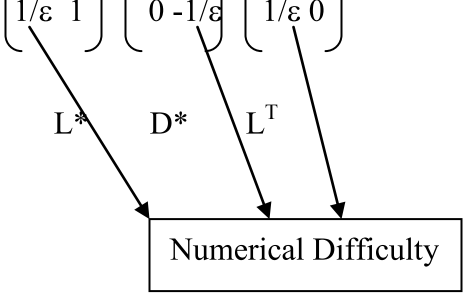
Solver02 + symbolic + … = solver00

Suppose $[A] = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & -1/\varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\varepsilon & 0 \end{pmatrix}^T$

λ = E values =

So $\lambda_1 = +$

$\lambda_2 = -$



Comparisons of piertwo180c.f And Boeing Indefinite Solvers

Neq	Ncoff	Sum(displ.)	Max Displ.	CPU Seconds (Cray-YMP)	Rel. Err
51	218	2.27E-02	2.00E-03	0.003	1.40E-13
(Boeing)		2.27E-02	2.00E-03	0.041	7.00E-14
247	2009	3.1596564	0.15253658	0.021	9.27E-10
(Boeing)		3.1596564	0.15253658	0.245	4.03E-10
1440	22137	29.68462	0.2028931	0.571	6.16E-10
(Boeing)		29.68462	0.20289318	2.351	3.26E-10
2430	75206	34.6802	9.31E-02	6.135	1.01E-11
(Boeing)		34.7033	9.31E-02	7.736	9.97E-11
7767	76111				
(Boeing)		113.713453	0.1610576	9.256	6.00E-08
15367	286044	512.35	0.205696	36.62	2.73E-09
(Boeing)		512.35	0.205696	35.77	4.38E-11

9.2 Substructure Displacement Analysis of a Two-Bay Truss

As an illustrative example of the substructure displacement analysis, a simple two-bay pin-jointed truss (Fig.9.6) will be analyzed. This cantilever truss is attached at one end to a rigid wall and is loaded by external forces P_1, P_2, and P_3 at the free end. The truss will be partitioned into two substructures by disconnecting the outer bay from the remainder of the structure. An exploded view of the two selected substructures is shown in Fig.9.7. Naturally, other choices for partitioning are possible also. For example, the center vertical member can be sliced vertically into two halves to form two substructures.

The first step in the substructure displacement analysis is the determination of substructure stiffness matrices, which are obtained by combining the element stiffness matrices in the datum system into stiffness matrices for assembled substructures using the summation procedure described in earlier sections.

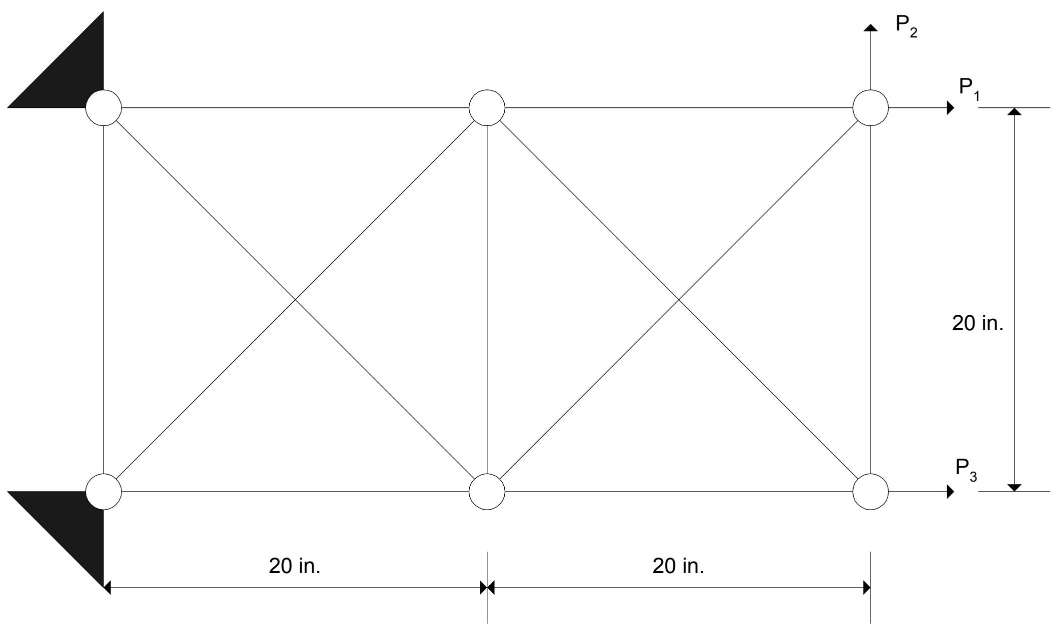


Fig. 9.6 Truss geometry and loading. Cross-sectional areas: vertical and horizontal bars 1.0 in^2; diagonal bars 0.707 $in.^2$ ($\sqrt{2}/2$ $in.^2$); E= $10*10^4$ $lb/in.^2$

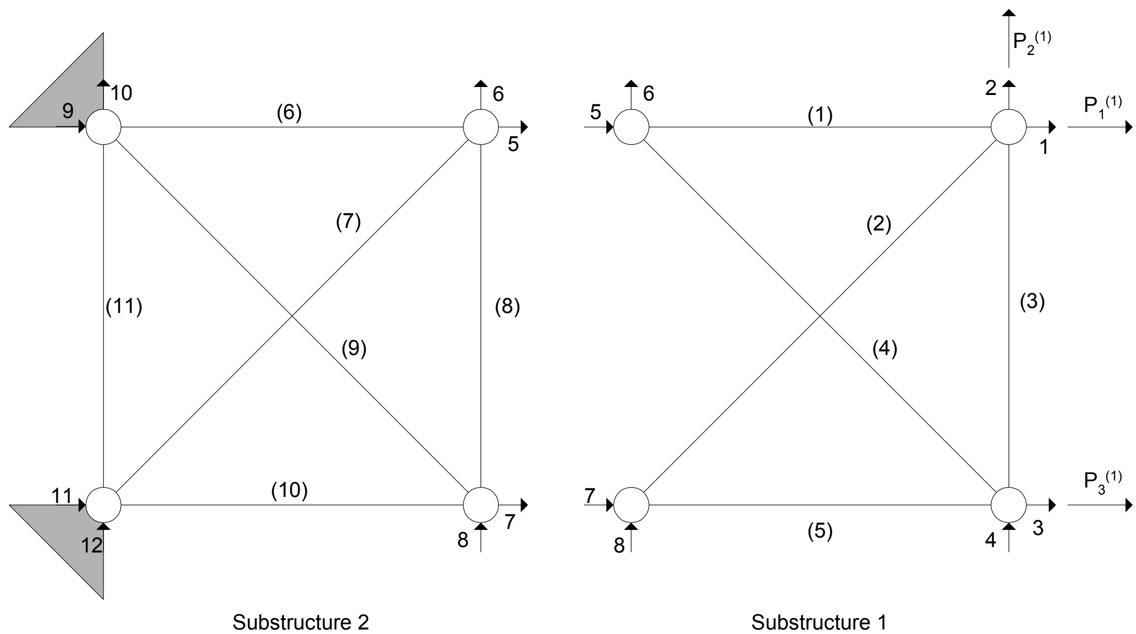


Fig. 9.7 Substructure partitioning.

$$
K^{(1)} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \\ \end{array}
\left(\begin{array}{cccccccc}
5 & & & & \text{Symmetric} & & & \\
1 & 5 & & & & & & \\
0 & 0 & 5 & & & & & \\
0 & -4 & -1 & 5 & & & & \\
-4 & 0 & -1 & 1 & 5 & & & \\
0 & 0 & 1 & -1 & -1 & 1 & & \\
-1 & -1 & -4 & 0 & 0 & 0 & 5 & \\
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array} \right) 0.125\text{x } 10^6
$$

$$
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}
$$

$$
K^{(2)} = \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\left(\begin{array}{cccccccc}
5 & & & & & & & \\
1 & 5 & & & & & & \\
0 & 0 & 5 & & & & & \\
0 & -4 & -1 & 5 & & & & \\
-4 & 0 & -1 & 1 & 5 & & & \\
0 & 0 & 1 & -1 & -1 & 5 & & \\
-1 & -1 & -4 & 0 & 0 & 0 & 5 & \\
-1 & -1 & 0 & 0 & 0 & -4 & 1 & 5 \\
\end{array} \right) 0.125\text{x } 10^6
$$

$$
\begin{array}{cccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}
$$

$K_b^{(1)} = K_{bb}^{(1)} - K_{bi}^{(1)}(K_{ii}^{(1)})^{-1}K_{ib}^{(1)}$

$$= \begin{array}{c|cccc|} & 5 & 6 & 7 & 8 \\ \hline 5 & 5 & -1 & 0 & 0 \\ 6 & -1 & 1 & 0 & 0 \\ 7 & 0 & 0 & 5 & 1 \\ 8 & 0 & 0 & 1 & 1 \end{array} \frac{10^6}{8} - \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 5 & -4 & 0 & -1 & 1 \\ 6 & 0 & 0 & 1 & -1 \\ 7 & -1 & -1 & -4 & 0 \\ 8 & -1 & -1 & 0 & 0 \end{array} \frac{10^6}{8}$$

$$\times \left\{ \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & -4 \\ 3 & 0 & 0 & 5 & -1 \\ 4 & 0 & -4 & -1 & 5 \end{array} \frac{10^6}{8} \right\}^{-1} \begin{array}{c|cccc|} & 5 & 6 & 7 & 8 \\ \hline 1 & -4 & 0 & -1 & -1 \\ 2 & 0 & 0 & -1 & -1 \\ 3 & -1 & 1 & -4 & 0 \\ 4 & 1 & -1 & 0 & 0 \end{array} \frac{10^6}{8}$$

$$= \begin{array}{c|cccc|} & 5 & 6 & 7 & 8 \\ \hline 5 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 & -1 \\ 7 & 0 & 0 & 0 & 0 \\ 8 & 0 & -1 & 0 & 1 \end{array} \frac{10^6}{22}$$

Displacements 1 to 4 represent the inerior displacements on substructure 1; however, there are no interior displacements on substructure 2, hence

$K_b^{(2)} = K^{(2)}$

Combining $K_b^{(1)}$ and $K_b^{(2)}$ matrices to form the boundary stiffness matrix for the entire structure and at the same time eliminating rows and columns 9 to 12, we obtain

$$K_b = \begin{array}{c|cccc|} & 5 & 6 & 7 & 8 \\ \hline 5 & 55 & 11 & 0 & 0 \\ 6 & 11 & 59 & 0 & -48 \\ 7 & 0 & 0 & 55 & -11 \\ 8 & 0 & -48 & -11 & 59 \end{array} \frac{10^6}{88}$$

To determine the resultant boundary forces S_b we use

$S_b = P_b - R_b$

$\quad = P_b - K_{bi}^{(1)} (K_{ii}^{(1)})^{-1} P_i^{(1)}$

Noting now that there are no external forces on the substructure boundaries we have that $P_b = 0$

Hence:

$$S_b = -\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 5 & -4 & 0 & -1 & 1 \\ 6 & 0 & 0 & 1 & -1 \\ 7 & -1 & -1 & -4 & 0 \\ 8 & -1 & -1 & 0 & 0 \\ \hline \end{array}\ \frac{10^6}{8}\left\{\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & -4 \\ 3 & 0 & 0 & 5 & -1 \\ 4 & 0 & -4 & -1 & 5 \\ \hline \end{array}\ \frac{10^6}{8}\right\}^{-1}\begin{array}{c|c|} 1 & P_1^{(1)} \\ 2 & P_2^{(1)} \\ 3 & P_3^{(1)} \\ 4 & 0 \end{array}$$

$$= \begin{array}{c|cccc|} 5 & 11 & -11 & 0 & -11 \\ 6 & -1 & 5 & -1 & 6 \\ 7 & 0 & 11 & 11 & 11 \\ 8 & 1 & 6 & 1 & 5 \end{array}\ \frac{1}{11}\begin{bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \\ 0 \end{bmatrix} = \begin{array}{c|ccc|} 5 & 11 & -11 & 0 \\ 6 & -1 & 5 & -1 \\ 7 & 0 & 11 & 11 \\ 8 & 1 & 6 & 1 \end{array}\ \frac{1}{11}\begin{bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{bmatrix}$$

The boundary displacements U_b are determined from

$$U_b = (K_b)^{-1} S_b$$

$$= \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\left[\begin{array}{cccc} 5 & 6 & 7 & 8 \\ \hline 55 & 11 & 0 & 0 \\ 11 & 59 & 0 & -48 \\ 0 & 0 & 55 & -11 \\ 0 & -48 & -11 & 59 \end{array}\cdot\frac{10^6}{88}\right]^{-1}\cdot\begin{pmatrix} 11 & -11 & 0 \\ -1 & 5 & -1 \\ 0 & 11 & 11 \\ 1 & 6 & 1 \end{pmatrix}\cdot\frac{1}{11}\cdot\begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

$$= \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{array}{cccc} 5 & 6 & 7 & 8 \\ \hline 119 & -71 & -12 & -60 \\ -71 & 355 & 60 & 300 \\ -12 & 60 & 119 & 71 \\ -60 & 300 & 71 & 355 \end{array}\cdot\frac{2\cdot 10^{-6}}{131}\cdot\begin{pmatrix} 11 & -11 & 0 \\ -1 & 5 & -1 \\ 0 & 11 & 11 \\ 1 & 6 & 1 \end{pmatrix}\cdot\frac{1}{11}\cdot\begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

$$= \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{pmatrix} 120 & -196 & -11 \\ -76 & 456 & 55 \\ -11 & 197 & 120 \\ -55 & 461 & 76 \end{pmatrix}\cdot\frac{2\cdot 10^{-6}}{131}\cdot\begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

$$= \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}\begin{pmatrix} 1.832 & -2.992 & -0.168 \\ -1.160 & 6.962 & 0.840 \\ -0.168 & 3.008 & 1.832 \\ -0.840 & 7.038 & 1.160 \end{pmatrix}\cdot 10^{-6}\cdot\begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

While the interior displacements are calculated from
$U_i^{(1)} = (K_{ii}^{(1)})^{-1}P_i^{(1)} - (K_{ii}^{(1)})^{-1}K_{ib}^{(1)}U_b^{(1)}$

Now

$$(K_{ii}^{(1)})^{-1}P_i^{(1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left[\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 5 & 1 & 0 & 0 \\ 1 & 5 & 0 & -4 \\ 0 & 0 & 5 & -1 \\ 0 & -4 & -1 & 5 \end{pmatrix} \end{array} \cdot \frac{10^6}{8} \right]^{-1} \cdot \begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \\ 0 \end{pmatrix}$$

$$= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 10 & -6 & -1 & -5 \\ -6 & 30 & 5 & 25 \\ -1 & 5 & 10 & 6 \\ -5 & 25 & 6 & 30 \end{pmatrix} \cdot \frac{2 \cdot 10^{-6}}{11} \cdot \begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \\ 0 \end{pmatrix}$$

$$= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 10 & -6 & -1 \\ -6 & 30 & 5 \\ -1 & 5 & 10 \\ -5 & 25 & 6 \end{pmatrix} \cdot \frac{2 \cdot 10^{-6}}{11} \cdot \begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

And

$(K_{ii}^{(1)})^{-1}K_{ib}^{(1)}U_b^{(1)}$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 10 & -6 & -1 & -5 \\ -6 & 30 & 5 & 25 \\ -1 & 5 & 10 & 6 \\ -5 & 25 & 6 & 30 \end{pmatrix} \end{array} \cdot \frac{2 \cdot 10^{-6}}{11} \cdot \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} -4 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \end{array} \cdot \frac{10^6}{8}$$

$$\times \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 120 & -196 & -11 \\ -76 & 456 & 55 \\ -11 & 197 & 120 \\ -55 & 461 & 76 \end{pmatrix} \cdot \frac{2 \cdot 10^{-6}}{131} \cdot \begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

$$= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} -1341 & 2151 & 100 \\ 2151 & -9369 & -2172 \\ 100 & -2172 & -1341 \\ 2172 & -9364 & -2151 \end{pmatrix} \cdot \frac{2 \cdot 10^{-6}}{11 \cdot 131} \cdot \begin{pmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \end{pmatrix}$$

Hence

$$U_i^{(1)} = \begin{matrix}1\\2\\3\\4\end{matrix}\left[\begin{pmatrix}10 & -6 & -1\\ -6 & 30 & 5\\ -1 & 5 & 10\\ -5 & 25 & 6\end{pmatrix}\cdot\frac{\left(2\cdot10^{-6}\right)}{11} - \begin{pmatrix}-1341 & 2151 & 100\\ 2151 & -9369 & -2172\\ 100 & -2172 & -1341\\ 2172 & -9364 & -2151\end{pmatrix}\cdot\frac{\left(2\cdot10^{-6}\right)}{11\cdot131}\right]\cdot\begin{pmatrix}P_1^{(1)}\\ P_2^{(1)}\\ P_3^{(1)}\end{pmatrix}$$

$$= \begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}241 & -267 & -21\\ -267 & 1209 & 257\\ -21 & 257 & 241\\ -257 & 1149 & 267\end{pmatrix}\cdot\frac{2\cdot10^{-6}}{131}\cdot\begin{pmatrix}P_1^{(1)}\\ P_2^{(1)}\\ P_3^{(1)}\end{pmatrix}$$

$$= \begin{matrix}1\\2\\3\\4\end{matrix}\begin{pmatrix}3.679 & -4.076 & -0.321\\ -4.076 & 18.458 & 3.924\\ -0.321 & 3.924 & 3.679\\ -3.924 & 17.542 & 4.076\end{pmatrix}\cdot10^{-6}\cdot\begin{pmatrix}P_1^{(1)}\\ P_2^{(1)}\\ P_3^{(1)}\end{pmatrix}$$

The matrices U_b and $U_i^{(1)}$ give all the required displacements. The element forces and stresses on individual elements can then be determined from the element stiffness matrices and the joint displacements.