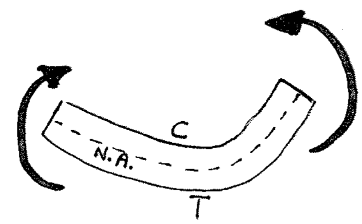
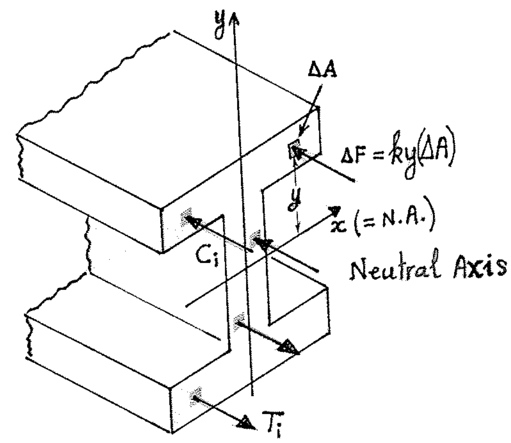
Chapter 9 : Moment of Inertia

Second Moment (or Moment of Inertia) of an Area

Pure Bending stress 



$$\sigma = \left(\frac{M}{I}\right) y = (k)\, y$$

$$\Delta F = \sigma(\Delta A) = ky(\Delta A)$$

$$R \equiv \text{Resultant force} \equiv \sum_i -C_i + T_i$$

$$-C_i + T_i \equiv R = \int ky\, dA = k \underbrace{\int y\, dA} = k\, \underbrace{\bar{y}}\, A = 0$$

zero, since the centroid of the beam's cross-section is on the N.A.

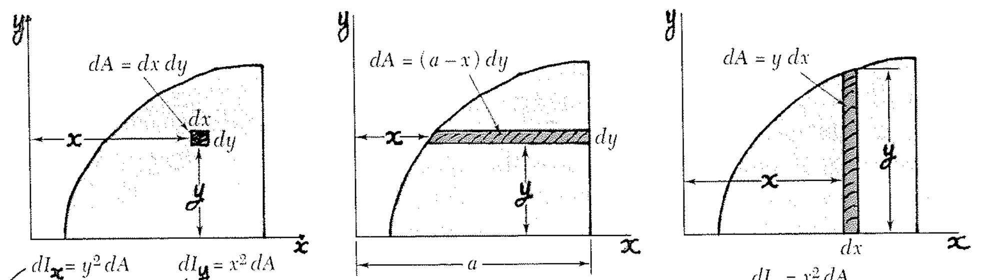
Since $R = 0 \Rightarrow$ system of forces ΔF reduces to a COUPLE $\left(\vec{C} \equiv \sum_i \vec{C}_i \text{ and } \vec{T} = \sum_i \vec{T}_i\right)$, about the x-axis

$$\Delta M_x = (\Delta F) * y = (ky * \Delta A) * y = ky^2 \Delta A$$

$$M_x = \int ky^2 dA = k \underbrace{\int y^2 dA}$$

$\boxed{\equiv I_x = \text{Moment of Inertia of the area with respect to } x\text{-axis}}$

How to Compute I_x (and/or I_y) of a General Cross-Section?



$dI_x = y^2\,dA$ $\quad$ $dI_y = x^2\,dA$

$I_x = \iint y^2\,dA = \iint y^2\,dx\,dy$ $\quad$ (a) $\quad$ $I_y = \iint x^2\,dA = \iint x^2\,dx\,dy$

$dI_x = y^2\,dA$

(b)

$I_x = \iint y^2\,dA = \int y^2 (a-x)\,dy$

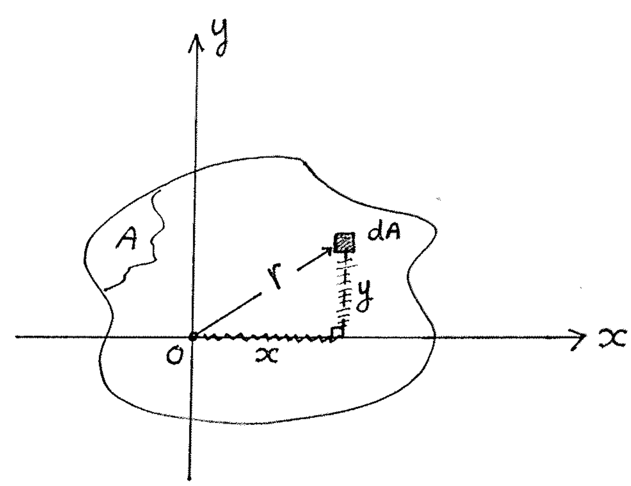
$dI_y = x^2\,dA$

(c)

$I_y = \iint x^2\,dA$

$I_y = \int x^2\,y\,dx$

Polar Moment of Inertia



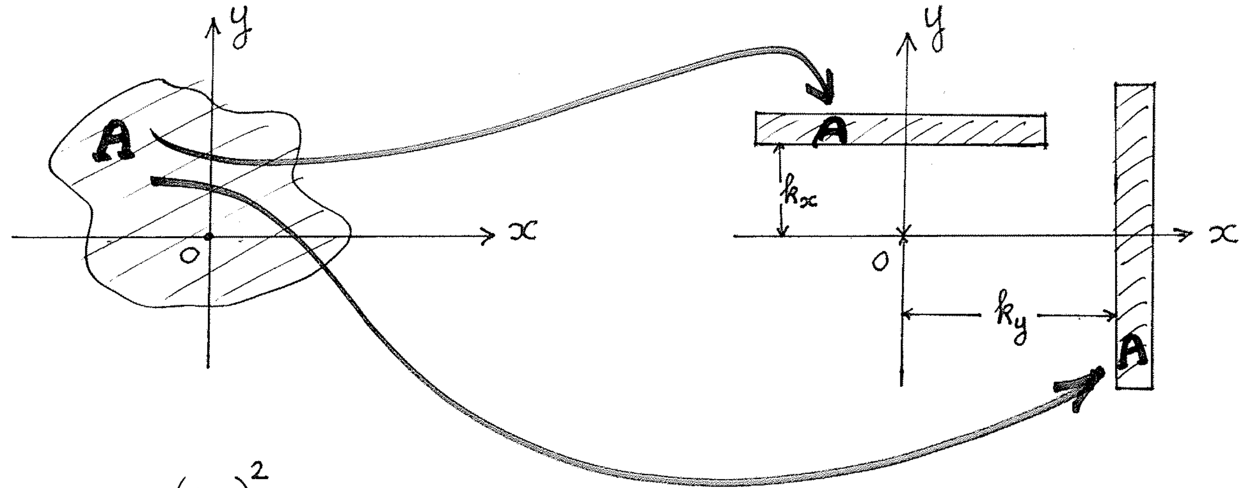
$$J_o \equiv \int r^2\,dA$$

$$J_o = \int (x^2+y^2)\,dA$$

$$J_o = \int x^2\,dA + \int y^2\,dA$$

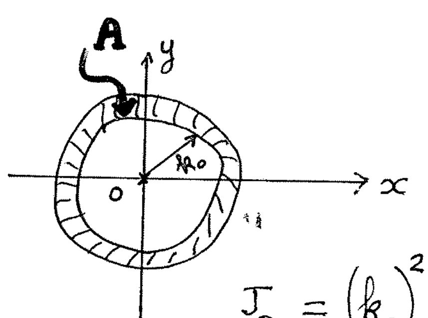
$$\boxed{J_o = I_y + I_x} \quad \ldots\ldots \ (9.4)$$

Radius of Gyration of an Area



$$I_x = (k_x)^2 A \implies k_x = \sqrt{\frac{I_x}{A}} \qquad (9.5)$$

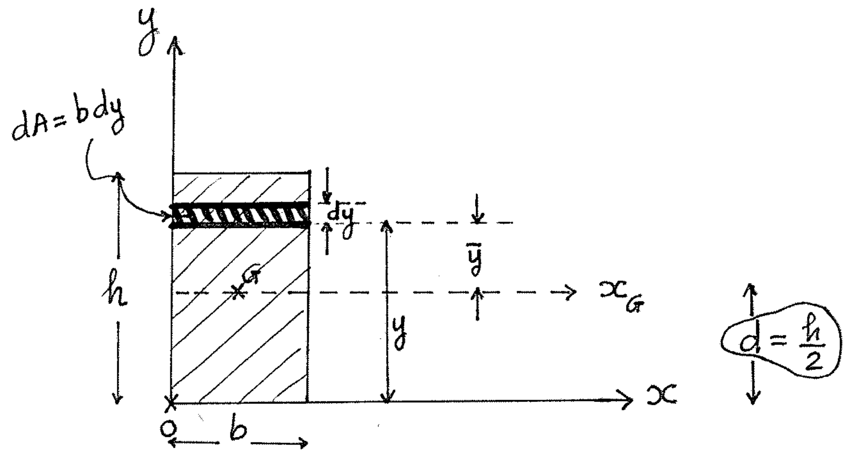
$$I_y = (k_y)^2 A \implies k_y = \sqrt{\frac{I_y}{A}} \qquad (9.6)$$



$$J_o = (k_o)^2 A \implies k_o = \sqrt{\frac{J_o}{A}} \qquad (9.7)$$

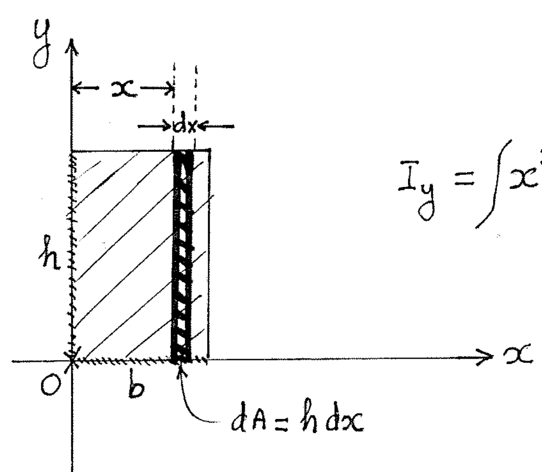
Note: $$k_o^2 = \frac{J_o}{A} = \frac{I_x + I_y}{A} = (k_x)^2 + (k_y)^2 \qquad (9.8)$$

Moment of Inertia of a Rectangular Area



$$I_x = \int y^2 \, dA = \int_0^h y^2 \, b \, dy = b\left[\frac{y^3}{3}\right]_0^h = \boxed{\frac{bh^3}{3} = I_x}$$

$$I_{x_G} = \int \bar{y}^2 \, dA = \int_{-h/2}^{h/2} \bar{y}^2 \, b \, d\bar{y} = b\left[\frac{\bar{y}^3}{3}\right]_{-h/2}^{+h/2} = b\left\{\left[\frac{h^3}{24}\right]-\left[\frac{-h^3}{24}\right]\right\} = \boxed{\frac{bh^3}{12} = I_{x_G}}$$



$$I_y = \int x^2 \, dA = \int_0^b x^2 \, h \, dx = h\left[\frac{x^3}{3}\right]_0^b = \boxed{\frac{hb^3}{3} = I_y}$$

Parallel Axis Theorem

verify (rectangular area)

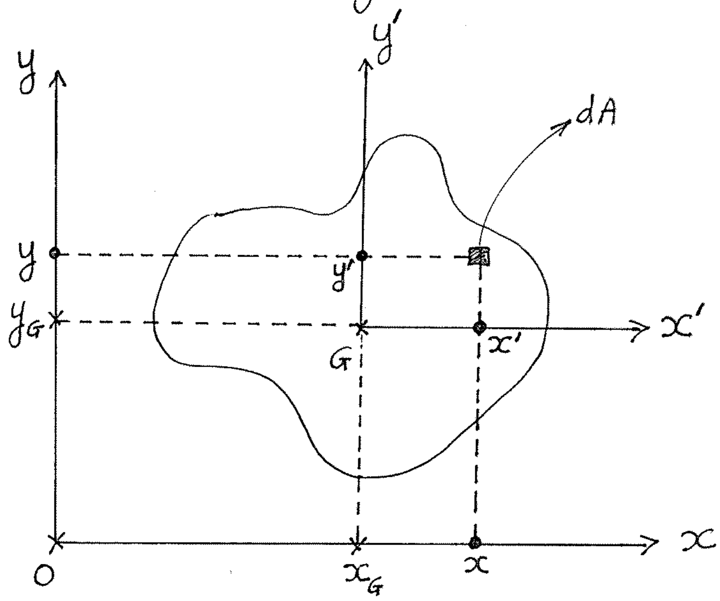
$$\frac{bh^3}{3} \underset{\text{YES}}{\overset{??}{=}} \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2$$

$$\boxed{I_x = I_{x_G} + A(d)^2}$$

where:

$d \equiv$ perpendicular distance between x & x_G axis

Product of Inertia



Let $G \equiv$ centroid of 2-D object

$$I_{xy} = \int xy \, dA$$

$$I_{xy} = \int (x_G + x')(y_G + y') \, dA$$

$$I_{xy} = \int x_G y_G \, dA + \int x_G y' \, dA + \int x' y_G \, dA + \int x' y' \, dA$$

$$I_{xy} = x_G y_G A + \text{zero} + \text{zero} + I_{x'y'}$$

$$\boxed{I_{xy} = I_{x'y'} + A(x_G y_G)} \longleftrightarrow \boxed{\text{very similar to parallel axis theorem!}}$$

Notes:

(a) x_G, y_G = constants

(b) Since G is the centroid of 2-D object, when integral over entire area then x', y' represent/become the centroid location (with respect to x', y' axis).

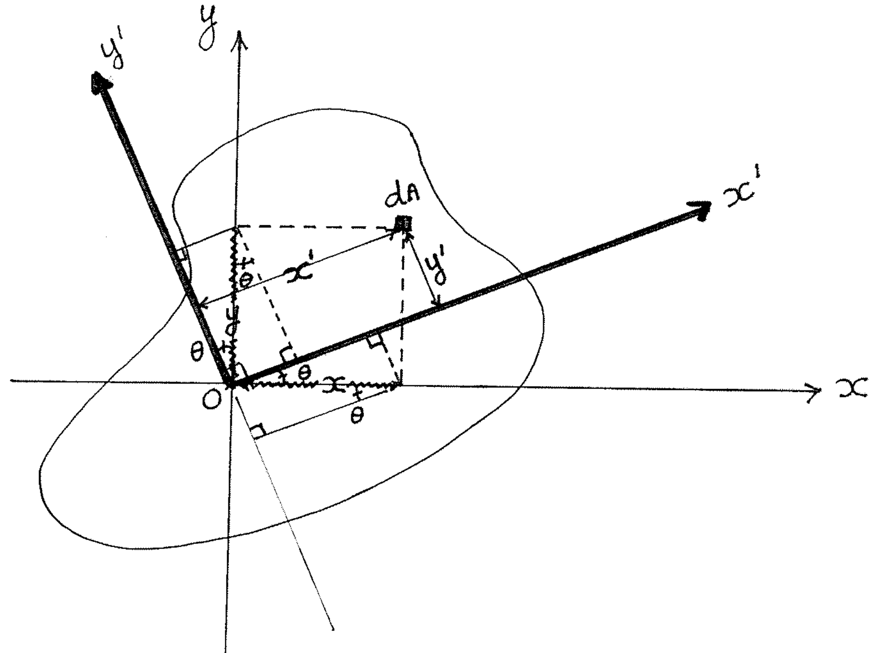
Thus $x' \longrightarrow x_G = 0$ (wrt x', y' axis)

$y' \longrightarrow y_G = 0$ (wrt x', y' axis)

Hence $\int x_G y' \, dA = x_G \underbrace{\int y' \, dA}_{\text{zero}} = 0$

Similarly $\int y_G x' \, dA = 0$

Principal Moment of Inertia, MOHR Circle



The projection of the x, y coordinates of an infinitesimal area dA onto the x' axis gives:

$$x' = x\cos\theta + y\sin\theta \qquad (1)$$

Similary:

$$y' = y\cos\theta - x\sin\theta \qquad (2)$$

Recalled:

$$I_{x'} = \int (y')^2\, dA = \int (y\cos\theta - x\sin\theta)^2\, dA \qquad (3)$$

$$I_{y'} = \int (x')^2\, dA = \int (x\cos\theta + y\sin\theta)^2\, dA \qquad (4)$$

$$I_{x'y'} = \int x'y'\, dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA \qquad (5)$$

Expanding the RHS of Eqs. (3-5), and using the formulas $I_x = \int y^2 \, dA$; $I_y = \int x^2 \, dA$; $I_{xy} = \int xy \, dA$, one gets:

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos(2\theta) - I_{xy} \sin(2\theta) \quad \text{———} \quad (6)$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos(2\theta) + I_{xy} \sin(2\theta) \quad \text{———} \quad (7)$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin(2\theta) + I_{xy} \cos(2\theta) \quad \text{———} \quad (8)$$

From Eqs. (6 & 7), one obtains:

$$I_{x'} + I_{y'} = I_x + I_y \quad \text{———} \quad (9)$$

To obtain Mohr circle equation, from Eq. (6), one gets:

$$\left[I_{x'} - \frac{I_x + I_y}{2}\right]^2 = \left[\frac{I_x - I_y}{2} \cos(2\theta) - I_{xy} \sin(2\theta)\right]^2 \quad \text{———} \quad (10)$$

From Eq. (8), one gets:

$$\left[I_{x'y'}\right]^2 = \left[\frac{I_x - I_y}{2} \sin(2\theta) + I_{xy} \cos(2\theta)\right]^2 \quad \text{———} \quad (11)$$

Adding Eqs. (10, 11), sides by sides, one obtains:

$$\left[I_{x'} - \frac{I_x + I_y}{2}\right]^2 + \left[I_{x'y'}\right]^2 = \left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2 \quad \text{———} \quad (12)$$

Define :

$$I_{Ave} \equiv \frac{I_x + I_y}{2} \qquad (13)$$

$$R \equiv \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2} \qquad (14)$$

Then, Eq. (12) becomes a (Prof. Mohr) Circle Equation :

$$\left(I_{x'} - I_{Ave}\right)^2 + \left(I_{x'y'}\right)^2 = (R)^2 \qquad (15)$$