Chapter 03.06
False-Position Method of Solving a Nonlinear Equation.

Introduction

In chapter 03.03, the Bisection method has been introduced and explained as one of the simple "bracketing methods" for solving a nonlinear equation, which can be expressed in the general form

$$f(x) = 0 \tag{1}$$

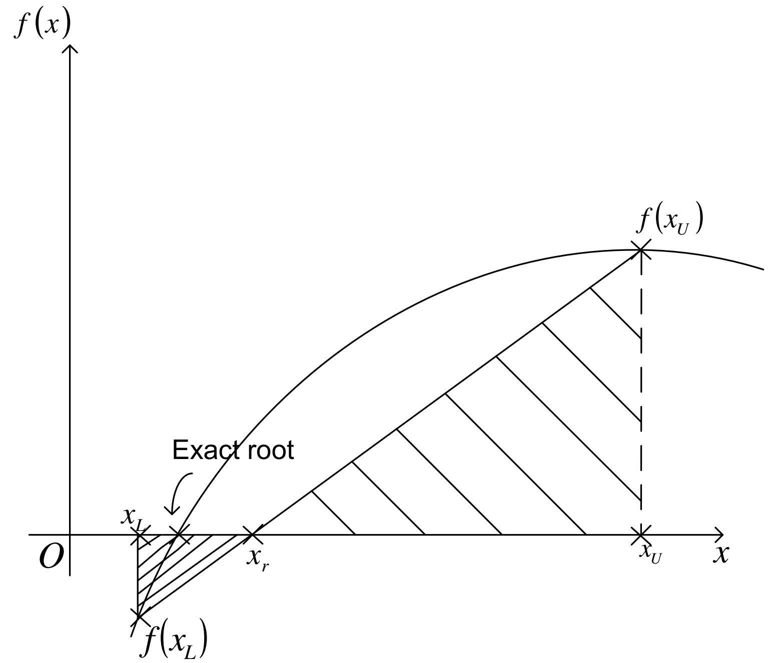

Figure 1 False-Position Method

The above nonlinear equation can be stated as finding the value of x (or the root's value) such that Equation (1) is satisfied. In the Bisection method, having identified the proper values of x_L (lower bound value) and x_U (upper bound value) for the current bracket, such that

$$f(x_L) * f(x_U) < 0 \tag{2}$$

The next predicted/improved root $(= x_r)$ can be computed as the mid-point between x_L and x_U. Thus, one computes:

$$x_r = \frac{x_L + x_U}{2} \tag{3}$$

The new bound or new upper bound is then established, and the procedure is repeated until the convergence is achieved (such as the new lower and upper bounds are sufficiently close to each other).

However, in the situation shown in Figure 1, the Bisection method may not be efficient, since it does not take into consideration that $f(x_L)$ is much closer to zero as compared to $f(x_U)$. In other words, the next predicted root x_r should be closer to x_L (in this particular example, shown in Figure 1), and should NOT be at the mid-point of x_L and x_U.

False-Position Method

Based on two similar triangles, shown in Figure 1, one gets:

$$\frac{f(x_L)}{x_r - x_L} = \frac{f(x_U)}{x_r - x_U} \tag{4}$$

The signs for both sides of Equation (4) is consistent, since:

$$f(x_L) < 0;\ x_r - x_L > 0$$
$$f(x_U) > 0;\ x_r - x_U < 0$$

From Equation (4), one obtains

$$(x_r - x_L)f(x_U) = (x_r - x_U)f(x_L)$$
$$x_U f(x_L) - x_L f(x_U) = x_r\{f(x_L) - f(x_U)\}$$

The above equation can be solved to obtain the next predicted root x_r, as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \tag{5}$$

The above equation, through simple algebraic manipulations, can also be expressed as:

$$x_r = x_U - \frac{f(x_U)\{x_L - x_U\}}{f(x_L) - f(x_U)} \tag{6}$$

or

$$x_r = x_L - \frac{f(x_L)}{\left\{\dfrac{f(x_U) - f(x_L)}{x_U - x_L}\right\}} \tag{7}$$

Step-by-Step False-Position Algorithms

The steps to apply the False-Position method to find the root of the equation $f(x) = 0$ are

1. Choose x_L and x_U as two guesses for the root such that $f(x_L)f(x_U) < 0$, or in other words, $f(x) = 0$ changes sign between x_L and x_U.
2. Estimate the root, x_m of the equation $f(x) = 0$ as $x_m = \dfrac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$
3. Now check the following
   a) If $f(x_L)f(x_m) < 0$, then the root lies between x_L and x_m; then $x_L = x_L$ and $x_U = x_m$.

b) If $f(x_L)f(x_m) > 0$, then the root lies between x_m and x_U; then $x_L = x_m$ and $x_U = x_U$.

c) If $f(x_L)f(x_m) = 0$, then the root is x_m. Stop the algorithm if this is true.

4. Find the new estimate of the root

$$x_m = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

Find the absolute relative approximate error as

$$|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

where

x_m^{new} = estimated root from present iteration

x_m^{old} = estimated root from previous iteration

5. Compare the absolute relative approximate error $|\in_a|$ with the pre-specified relative error tolerance $\in_s$, *say* $\in_s = 10^{-3} = 0.001$. If $|\in_a| > \in_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Notes: The False-Position and Bisection algorithms are quite similar. The only difference is the formula used to calculate the new estimate of the root x_m, shown in steps #2 and 4!

Example 1

You are working for “DOWN THE TOILET COMPANY” that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5cm. You are asked to find the depth to which the ball is submerged when floating in water.

The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the false-position method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the converged iteration.

Solution

From the physics of the problem, the ball would be submerged between $x = 0$ *and* $x = 2R$, where R = radius of the ball,

that is

$$0 \le x \le 2R$$

$$0 \le x \le 2(0.055)$$

$$0 \le x \le 0.11$$

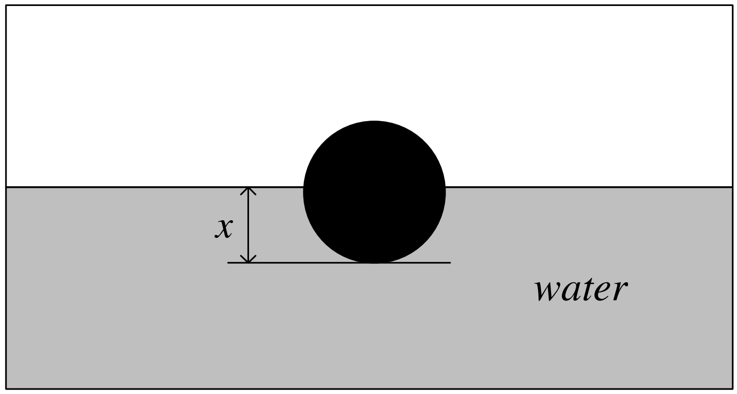

Figure 2 Floating ball problem

Let us assume

$x_L = 0,\ x_U = 0.11$

Check if the function changes sign between x_L and x_U

$$f(x_L) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$
$$f(x_U) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_L)f(x_U) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

So there is at least one root between x_L and x_U, that is between 0 and 0.11.

Iteration 1

The estimate of the root is

$$\begin{aligned} x_m &= \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \\ &= \frac{0.11\times3.993\times10^{-4} - 0\times(-2.662\times10^{-4})}{3.993\times10^{-4} - (-2.662\times10^{-4})} \\ &= 0.0660 \end{aligned}$$

$$f(x_m) = f(0.0660) = (0.0660)^3 - 0.165(0.0660)^2 + (3.993\times10^{-4}) = -3.1944\times10^{-5}$$
$$f(x_L)f(x_m) = f(0)f(0.0660) = (+)(-) < 0$$

Hence the root is bracketed between x_L and x_m, that is, between 0 and 0.066. So, the lower and upper limit of the new bracket is

$x_L = 0,\ x_U = 0.0660$

<u>Iteration 2</u>

$$x_m = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$
$$= \frac{0.0660 \times 3.993 \times 10^{-4} - 0 \times (-3.1944 \times 10^{-5})}{3.993 \times 10^{-4} - (-3.1944 \times 10^{-5})}$$
$$= 0.0611$$

$$f(x_m) = f(0.0611) = (0.0611)^3 - 0.165(0.0611)^2 + (3.993 \times 10^{-4}) = 1.1320 \times 10^{-5}$$

$$f(x_L) f(x_m) = f(0) f(0.0611) = (+)(+) > 0$$

Hence, the lower and upper limit of the new bracket is

$$x_L = 0.0611,\ x_U = 0.0660$$

$$\in_a = \left| \frac{0.0611 - 0.0660}{0.0611} \right| \times 100 \cong 8\%$$

<u>Iteration 3</u>

$$x_m = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$
$$= \frac{0.0660 \times 1.132 \times 10^{-5} - 0.0611 \times (-3.1944 \times 10^{-5})}{1.132 \times 10^{-5} - (-3.1944 \times 10^{-5})}$$
$$= 0.0624$$

$$f(x_m) = -1.1313 \times 10^{-7}$$

$$f(x_L) f(x_m) = f(0.0611) f(0.0624) = (+)(-) < 0$$

Hence, the lower and upper limit of the new bracket is

$$x_L = 0.0611,\ x_U = 0.0624$$

$$\in_a = \left| \frac{0.0624 - 0.0611}{0.0624} \right| \times 100 \cong 2.05\%$$

All iterations' results are summarized in the following Table 1

Iteration	x_L	x_U	x_m	$\left|\in_a\right|\%$	$f(x_m)$
1	0.0000	0.1100	0.0660	N/A	$-3.1944\text{x}10^{-5}$
2	0.0000	0.0660	0.0611	8.00	$1.1320\text{x}10^{-5}$
3	0.0611	0.0660	0.0624	2.05	$-1.1313\text{x}10^{-7}$
4	0.0611	0.0624	0.0632377619	0.02	$-3.3471\text{x}10^{-10}$

Table 1 Root of $f(x)=x^3-0.165x^2+3.993\times10^{-4}=0$ for False-Position Method.

Hence:

$$\left|\in_a\right| \le 0.5\times10^{2-m}$$
$$0.02 \le 0.5\times10^{2-m}$$
$$0.04 \le 10^{2-m}$$

$$\log(0.04) \le 2-m$$
$$m \le 2-\log(0.04)$$
$$m \le 2-(-1.3979)$$
$$m \le 3.3979$$
$$So,\ m=3$$

The number of significant digits at least correct in the estimated root of 0.062377619 at the end of 4^{th} iteration is 3.

<u>Example 2</u>

Find the root of $f(x)=(x-4)^2(x+2)=0$, using the initial guess $x_L=-2.5,\ x_U=-1.0,\ \text{and}\in_s=10^{-3}=0.001$.

Solution

The iterations' results can be summarized in Table 2.

Iteration	x_L	x_U	$f(x_L)$	$f(x_U)$	x_m	$\lvert\in_a\rvert\%$	$f(x_m)$
1	-2.5	-1	-21.125	25	-1.813	N/A	6.319
2	-2.5	-1.813	-21.125	6.319	-1.971	8.02	1.0275
3	-2.5	-1.9712	-21.125	1.0275	-1.9957	1.23	0.15421
4	-2.5	-1.9957	-21.125	0.15421	-1.9994	0.1828	0.02287
5	-2.5	-1.9994	-21.125	0.02287	-1.9999	0.027061	0.003383

Table 2 Root of $f(x)=(x-4)^2(x+2)=0$ for False-Position Method.

Hence:

$$\lvert\in_a\rvert \le 0.5\times 10^{2-m}$$
$$0.027061 \le 0.5\times 10^{2-m}$$
$$0.05412 \le 10^{2-m}$$

$$\log(0.05412) \le 2-m$$
$$-1.2666 \le 2-m$$
$$m \le 3.2666$$

= at least 3 significant digits are accurate can be expected from the converged result.

Multiple Choice Tests

1. The False-Position method for finding roots of nonlinear equations belongs to a class of a (an) _____________ method.

(a) Open
(b) Bracketting
(c) Random
(d) Graphical

2. The newly predicted root for False-Position, and Secant methods can be respectively given as

$$x_r = x_U - \frac{f(x_U)\{x_U - x_L\}}{f(x_U) - f(x_L)} \qquad \text{(6, repeated)}$$

and

$x_{i+1} = x_i - \dfrac{f(x_i)\{x_i - x_{i-1}\}}{f(x_i) - f(x_{i-1})}$, as shown in Equation (3) of Chapter 3.05. While the appearance of the above 2 equations look essentially identical, and both methods do require 2 initial guesses, the major difference between the above two formulas is

(a) False-Position method is <u>NOT</u> guarantee to converge.
(b) Secant method is guarantee to converge
(c) Secant method requires the 2 initial guesses x_{i-1} *and* x_i to satisfy $f(x_{i-1}) \times f(x_i) < 0$
(d) False-Position method requires the 2 initial guesses x_L *and* x_U to satisfy $f(x_L) \times f(x_U) < 0$

3. Given the following nonlinear equation

$f(x) = e^{-2x} + 4x^2 - 36 = 0$ and the two initial guesses (say, at iteration 1) $x_L = 1, x_U = 4$ (with $\in_s = 10^{-3}$ specified, relative approximated error tolerance).

Using the False-Position method, which of the following tables is correct (x_r = predicted root)?

(a)

Iteration	x_L	x_U	x_r
1	1	4	?
2	?	?	2.9387

(b)

Iteration	x_L	x_U	x_r
1	1	4	?
2	?	?	3.8792

(c)

Iteration	x_L	x_U	x_r
1	1	4	?
2	?	?	7.9238

(d)

Iteration	x_L	x_U	x_r
1	1	4	?
2	?	?	2.7839

4. Using the same data as stated in problem 4.3, which of the following table is correct?
($\in_a$ = approximated relative error, expressed in terms of percentage. Thus, if $\in_a$ = say, 1.27, then it means $\in_a$ = 1.27%)

(a)

Iteration	x_L	x_U	x_r	$\in_a$
1	1	4	?	?
2	?	?	?	11.63415

(b)

Iteration	x_L	x_U	x_r	$\in_a$
1	1	4	?	?
2	?	?	?	6.11354

(c)

Iteration	x_L	x_U	x_r	$\in_a$
1	1	4	?	?
2	?	?	?	5.14361

(d)

Iteration	x_L	x_U	x_r	$\in_a$
1	1	4	?	?
2	?	?	?	4.15361

5. Find the root of $f(x)=(x-4)^2(x+2)=0$, using the initial guess x_L=-2.5, x_U=-1.0, and $\in_s=10^{-6}$. The final, converged root found by the False-Position method is x_r=-1.9999997, and the corresponding approximated relative error $\in_a$=8.7610979 x 10^{-5}%. Based on these given information, the number of significant digits (of x_r) that can be trusted at least.

(a) 3 significant digits
(b) 4 significant digits
(c) 5 significant digits
(d) 6 significant digits

6. Given the nonlinear equation $f(x)=x^2-7.4x+13.69=0$, the False-Position method may have difficulty to find the root of this nonlinear equation, because.

(a) *f(x)* is a quadratic equation
(b) $\dfrac{df}{dx} \equiv f' =$ a general straight line
(c) Initial guess x_L and x_U can't satisfy $f(x_L)\times f(x_U)<0$
(d) The root can't be found, even by Newton-Raphson (NR) method.

References

[1] S.C. Chapra, R.P.Canale, Numerical Methods for Engineers, Fourth Edition, Mc-Graw Hill.

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