

Chapter 04.09

Cholesky and LDL^T Decomposition

Introduction

Solving large (and sparse) system of simultaneous linear equations (SLE) has been (and continues to be) a major challenging problem for many real-world engineering/science applications [1-2]. In matrix notation, the SLE can be represented as:

$$[A][x] = [b] \quad (1)$$

where $[A]$ = known coefficient matrix, with dimension $n \times n$

$[b]$ = known right-hand-side (RHS) $n \times 1$ vector

$[x]$ = unknown $n \times 1$ vector.

Symmetrical Positive Definite (SPD) SLE

For many practical SLE, the coefficient matrix $[A]$ (see Equation (1)) is Symmetric Positive Definite (SPD). In this case, the efficient 3-step Cholesky algorithms [1-2] can be used.

A matrix $[A]_{n \times n}$ can be considered as SPD if either of the following conditions is satisfied:

- (a) If each and every determinant of sub-matrix A_{ii} ($i = 1, 2, \dots, n$) is positive, or..
- (b) If $y^T A y > 0$, for any given vector $[y]_{n \times 1} \neq \vec{0}$

As a quick example, let us make a test a test to see if the given matrix

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ is SPD?}$$

Based on criteria (a):

The given 3×3 matrix $[A]$ is symmetrical, because $a_{ij} = a_{ji}$. Furthermore, one has

$$\det[A]_{1 \times 1} = |2| = 2 > 0$$

$$\begin{aligned} \det[A]_{2 \times 2} &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 3 > 0 \end{aligned}$$

$$\begin{aligned} \det[A]_{3 \times 3} &= \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} \\ &= 1 > 0 \end{aligned}$$

Hence $[A]$ is SPD.

Based on criteria (b): For any given vector

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \neq \vec{0}, \text{ one computes}$$

$$\text{scalar} = y^T A y$$

$$\begin{aligned} &= [y_1 \ y_2 \ y_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= (2y_1^2 - 2y_1y_2 + 2y_2^2) + \{y_3^2 - 2y_2y_3\} \\ &= (y_1 - y_2)^2 + y_1^2 + y_2^2 + \{y_3^2 - 2y_2y_3\} \end{aligned}$$

$$\text{scalar} = (y_1 - y_2)^2 + y_1^2 + (y_2 - y_3)^2 > 0$$

Since the above scalar is always positive, hence matrix $[A]$ is SPD.

Step 1: Matrix Factorization phase

In this step, the coefficient matrix $[A]$ that is SPD can be decomposed (or factorized) into

$$[A] = [U]^T [U] \quad (2)$$

where $[U]$ is a $n \times n$ upper triangular matrix.

The following simple 3×3 matrix example will illustrate how to find the matrix $[U]$.

Various terms of the factorized matrix $[U]$ can be computed/derived as follows (see Equation (2)):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (3)$$

Multiplying two matrices on the right-hand-side (RHS) of Equation (3), then equating each upper-triangular RHS terms to the corresponding ones on the upper-triangular left-hand-side (LHS), one gets the following 6 equations for the 6 unknowns in the factorized matrix $[U]$.

$$u_{11} = \sqrt{a_{11}}; u_{12} = \frac{a_{12}}{u_{11}}; u_{13} = \frac{a_{13}}{u_{11}} \quad (4)$$

$$u_{22} = (a_{22} - u_{12}^2)^{\frac{1}{2}}; u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}}; u_{33} = (a_{33} - u_{13}^2 - u_{23}^2)^{\frac{1}{2}} \quad (5)$$

In general, for a $n \times n$ matrix, the diagonal and off-diagonal terms of the factorized matrix $[U]$ can be computed from the following formulas:

$$u_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} (u_{ki})^2 \right)^{\frac{1}{2}} \quad (6)$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}} \quad (7)$$

It is noted that if $i = j$, then the numerator of Equation (7) becomes identical to the terms under the square root in Equation (6). In other words, to factorize a general term u_{ij} , one simply needs to do the following steps:

Step 1.1: Compute the numerator of Equation (7), such as

$$Sum = a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}$$

Step 1.2 If u_{ij} is an off-diagonal term (say, $i < j$) then $u_{ij} = \frac{Sum}{u_{ii}}$ (See Equation (7)). Else, if u_{ij} is a diagonal term (that is, $i = j$), then $u_{ii} = \sqrt{Sum}$ (See Equation (6))

As a quick example, one computes:

$$u_{57} = \frac{a_{57} - u_{15}u_{17} - u_{25}u_{27} - u_{35}u_{37} - u_{45}u_{47}}{u_{55}} \quad (8)$$

Thus, for computing $u(i=5, j=7)$, one only needs to use the (already factorized) data in columns # $i (= 5)$, and # $j (= 7)$ of $[U]$, respectively.

Remarks

- In general, to find the (off-diagonal) factorized term u_{ij} , one only needs to utilize the “already factorized” columns # i , and # j information (see Figure 1). For example, if $i = 5$, and $j = 7$, then Figure 1 will lead to the same formula as shown earlier in Equation (7), or in Equation (8). Similarly, to find the (diagonal) factorized term u_{ii} , one simply needs to utilize columns # i , and # i (again!) information (see Figure 1). In this case, Figure 1 will lead to the same formula as shown earlier in Equation (6).

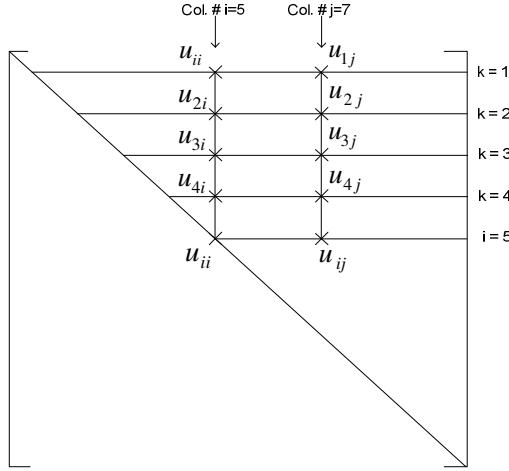


Figure 1 Cholesky Factorization for the term u_{ij}

- Since the square root operation involved during the Cholesky factorization phase (see Equation (6)), one must make sure the term under the square root is non-negative. This requirement satisfied by $[A]$ being SPD.

Step 2: Forward Solution phase

Substituting Equation (2) into Equation (1), one gets:

$$[U]^T [U] [x] = [b] \quad (9)$$

Let us define:

$$[U] [x] \equiv [y] \quad (10)$$

Then, Equation (9) becomes:

$$[U]^T [y] = [b] \quad (11)$$

Since $[U]^T$ is a lower triangular matrix, Equation (11) can be efficiently solved for the

intermediate unknown vector $[y]$, according to the order $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, hence the name “forward solution”.

As a quick example, one has (see Equation (11)):

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (12)$$

From the 1st row of Equation (12), one gets

$$\begin{aligned} u_{11}y_1 &= b_1 \\ y_1 &= \frac{b_1}{u_{11}} \end{aligned} \quad (13)$$

From the 2nd row of Equation (12), one gets

$$\begin{aligned} u_{12}y_1 + u_{22}y_2 &= b_2 \\ y_2 &= b_2 - \frac{u_{12}y_1}{u_{22}} \end{aligned} \quad (14)$$

Similarly

$$y_3 = \frac{b_3 - u_{13}y_1 - u_{23}y_2}{u_{33}} \quad (15)$$

In general, from the j^{th} row of Equation (12), one has

$$y_j = \frac{b_j - \sum_{i=1}^{j-1} u_{ij}y_i}{u_{jj}} \quad (16)$$

Step 3: Backward Solution phase

Since $[U]$ is an upper triangular matrix, Equation (10) can be efficiently solved for the original

unknown vector $[x]$, according to the order $\begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ \vdots \\ x_1 \end{bmatrix}$, hence the name “backward solution”.

As a quick example, one has (See Equation (10)):

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (17)$$

From the last (or $n^{\text{th}} = 3^{\text{rd}}$) row of Equation (17), one has

$$u_{33}x_3 = y_3.$$

hence

$$x_3 = \frac{y_3}{u_{33}} \quad (18)$$

Similarly:

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}} \quad (19)$$

and

$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}} \quad (20)$$

In general, one has:

$$x_j = \frac{y_j - \sum_{i=j+1}^n u_{ji}x_i}{u_{jj}} \quad (21)$$

Remarks

- (a) Amongst the above 3-step Cholesky algorithms, factorization phase in step 1 consumes about 95% of the total SLE solution time.
- (b) If the coefficient matrix $[A]$ is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following LDL^T factorized algorithms can be employed:

$$[A] = [L][D][L]^T \quad (22)$$

For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \quad (23)$$

Multiplying the three matrices on the RHS of Equation (23), then equating the resulting upper-triangular RHS terms of Equation (23) to the corresponding ones on the LHS, one obtains the following formulas for the “diagonal” $[D]$, and “lower-triangular” $[L]$ matrices:

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk} \quad (24)$$

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left(\frac{1}{d_{jj}} \right) \quad (25)$$

Thus, the LDL^T algorithms can be summarized by the following step-by-step procedures.

Step1: Factorization phase

$$[A] = [L][D][L]^T \quad (22, \text{ repeated})$$

Step 2: Forward solution and diagonal scaling phase

Substituting Equation (22) into Equation (1), one gets:

$$[L][D][L]^T [x] = [b] \quad (26)$$

Let us define:

$$[L]^T [x] = [y]$$

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (27)$$

$$x_i = y_i - \sum_{k=i+1}^n l_{ki} x_k; \text{ for } i = n, n-1, \dots, 1 \quad (28)$$

Also, define:

$$[D][y] = [z]$$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (29)$$

$$y_i = \frac{z_i}{d_{ii}}, \text{ for } i = 1, 2, 3, \dots, n \quad (30)$$

Then Equation (26) becomes:

$$[L][z] = [b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (31)$$

$$z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \quad \text{for } i = 1, 2, 3, \dots, n \quad (32)$$

Equation (31) can be efficiently solved for the vector $[z]$, then Equation (29) can be conveniently (and trivially) solved for the vector $[y]$.

Step 3: Backward solution phase

In this step, Equation (27) can be efficiently solved for the original unknown vector $[x]$.

Numerical Example 1 (Cholesky algorithms)

Solve the following SLE system for the unknown vector $[x]$?

$$[A][x] = [b]$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[b] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

The factorized, upper triangular matrix $[U]$ can be computed by either referring to Equations (6-7), or looking at Figure 1, as following:

$$\left. \begin{array}{l} u_{11} = \sqrt{a_{11}} \\ = \sqrt{2} \\ = 1.414 \\ u_{12} = \frac{a_{12}}{u_{11}} \\ = \frac{-1}{1.414} \\ = -0.7071 \\ u_{13} = \frac{a_{13}}{u_{11}} \\ = \frac{0}{1.414} \\ = 0 \end{array} \right\} \text{row 1 of } [U]$$

$$\left. \begin{aligned}
u_{22} &= \left\{ a_{22} - \sum_{k=1}^{i-1=1} (u_{ki})^2 \right\}^{\frac{1}{2}} \\
&= \left\{ 2 - (u_{12})^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{2 - (-0.7071)^2} \\
&= 1.225 \\
u_{23} &= \frac{a_{23} - \sum_{k=1}^{i-1=1} u_{ki} u_{kj}}{U_{22}} \\
&= \frac{-1 - u_{12} \times u_{13}}{1.225} \\
&= \frac{-1 - (-0.7071)(0)}{1.225} \\
&= -0.8165
\end{aligned} \right\} \text{row 2 of } [U]$$

$$\left. \begin{aligned}
u_{33} &= \left\{ a_{33} - \sum_{k=1}^{i-1=2} (u_{ki})^2 \right\}^{\frac{1}{2}} \\
&= \left\{ a_{33} - u_{13}^2 - u_{23}^2 \right\}^{\frac{1}{2}} \\
&= \sqrt{1 - (0)^2 - (-0.8165)^2} \\
&= 0.5774
\end{aligned} \right\} \text{row 3 of } [U]$$

Thus, the factorized matrix

$$[U] = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

The forward solution phase, shown in Equation (11), becomes:

$$[U]^T [y] = [b]$$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, Equation (16) can be used to solve:

$$y_1 = \frac{b_1}{u_{11}}$$

$$= \frac{1}{1.414}$$

$$= 0.7071$$

$$y_2 = \frac{b_2 - \sum_{i=1}^{j-1=1} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{12} = -0.7071)(y_1 = 0.7071)}{(u_{22} = 1.225)}$$

$$= 0.4082$$

$$y_3 = \frac{b_3 - \sum_{i=1}^{j-1=2} u_{ij} y_i}{u_{jj}}$$

$$= \frac{0 - (u_{13} = 0)(y_1 = 0.7071) - (u_{23} = -0.8165)(y_2 = 0.4082)}{(u_{33} = 0.5774)}$$

$$= 0.5774$$

The backward solution phase, shown in Equation (10), becomes:

$$[U][x] = [y]$$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

Thus, Equation (21) can be used to solve:

$$\begin{aligned}
x_3 &= \frac{y_j}{u_{jj}} \\
&= \frac{y_3}{u_{33}} \\
&= \frac{0.5774}{0.5774} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{y_j - \sum_{i=j+1=3}^{N=3} u_{ji} x_i}{u_{jj}} \\
&= \frac{y_2 - u_{23} x_3}{u_{22}} \\
&= \frac{0.4082 - (-0.8165)(1)}{1.225} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{y_j - \sum_{i=j+1=2}^{N=3} u_{ji} x_i}{u_{jj}} \\
&= \frac{y_1 - u_{12} x_2 - u_{13} x_3}{u_{11}} \\
&= \frac{0.7071 - (-0.7071)(1) - (0)(1)}{1.414} \\
&= 1
\end{aligned}$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Numerical Example 2 (LDL^T Algorithms)

Using the same data given in Numerical Example 1, find the unknown vector $[x]$ by LDL^T algorithms?

Solution:

The factorized matrices $[D]$ and $[L]$ can be computed from Equation (24), and Equation (25), respectively.

$$\left. \begin{array}{l}
d_{11} = a_{11} - \sum_{k=1}^{j-l=0} l_{jk}^2 d_{kk} \\
= a_{11} \\
= 2 \\
l_{11} = 1 \text{ (always !)} \\
l_{21} = \frac{a_{21} - \sum_{k=1}^{j-l=0} l_{ik} d_{kk} l_{jk}}{d_{jj}} \\
= \frac{a_{21}}{d_{11}} \\
= \frac{-1}{2} \\
= -0.5
\end{array} \right\} \text{Column 1 of matrices of } [D] \text{ and } [L]$$

$$\left. \begin{array}{l}
d_{22} = a_{22} - \sum_{k=1}^{j-l=1} l_{jk}^2 d_{kk} \\
= 2 - l_{21}^2 d_{11} \\
= 2 - (-0.5)^2 (2) \\
= 1.5 \\
l_{22} = 1 \text{ (always !)} \\
l_{32} = \frac{a_{32} - \sum_{k=1}^{j-l=1} l_{31} d_{11} l_{21}}{d_{22}} \\
= \frac{-1 - (0)(2)(-0.5)}{1.5} \\
= -0.6667
\end{array} \right\} \text{Column 2 of matrices } [D] \text{ and } [L]$$

$$\left. \begin{array}{l}
d_{33} = a_{33} - \sum_{k=1}^{j-l=2} l_{jk}^2 d_{kk} \\
= 1 - l_{31}^2 d_{11} - l_{32}^2 d_{22} \\
= 1 - (0)^2 (2) - (-0.6667)^2 (1.5) \\
= 0.3333
\end{array} \right\} \text{Column 3 of matrices } [D] \text{ and } [L]$$

Hence

$$[D] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix}$$

and

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix}$$

The forward solution shown in Equation (31), becomes:

$$[L][z] = [b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or,

$$z_i = b_i - \sum_{k=1}^{i-1} l_{ik} z_k \quad (32, \text{ repeated})$$

Hence

$$\begin{aligned} z_1 &= b_1 = 1 \\ z_2 &= b_2 - L_{21}z_1 \\ &= 0 - (-0.5)(1) \\ &= 0.5 \\ z_3 &= b_3 - L_{31}z_1 - L_{32}z_2 \\ &= 0 - (0)(1) - (-0.6667)(0.5) \\ &= 0.3333 \end{aligned}$$

The diagonal scaling phase, shown in Equation (29), becomes

$$[D][y] = [z]$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix}$$

or

$$y_i = \frac{z_i}{d_{ii}}$$

Hence

$$\begin{aligned} y_1 &= \frac{z_1}{d_{11}} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} y_2 &= \frac{z_2}{d_{22}} \\ &= \frac{0.5}{1.5} \\ &= 0.3333 \end{aligned}$$

$$\begin{aligned} y_3 &= \frac{z_3}{d_{33}} \\ &= \frac{0.3333}{0.3333} \\ &= 1 \end{aligned}$$

The backward solution phase can be found by referring to Equation (27)

$$[L]^T [x] = [y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

$$x_i = y_i - \sum_{k=i+1}^N l_{ki} x_k \quad (28, \text{repeated})$$

Hence

$$\begin{aligned}
 x_3 &= y_3 \\
 &= 1 \\
 x_2 &= y_2 - l_{32}x_3 \\
 &= 0.3333 - (-0.6667) \times 1 \\
 x_2 &= 1 \\
 x_1 &= y_1 - l_{21}x_2 - l_{31}x_3 \\
 x_1 &= 0.5 - (-0.5)(1) - (0)(1) \\
 &= 1
 \end{aligned}$$

Hence

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Remarks

Through this numerical example, one clearly sees that the “square root operations” have NOT been involved during the entire LDL^T algorithms. Thus, the coefficient matrix $[A]$, shown in Equation (1), is NOT required to be SPD.

Re-ordering Algorithms For Minimizing Fill-in Terms [1,2].

During the factorization phase (of Cholesky, or LDL^T algorithms), many “zero” terms in the original/given matrix $[A]$ will become “non-zero” terms in the factored matrix $[U]$. These new non-zero terms are often called as “fill-in” terms (indicated by the symbol F). It is, therefore, highly desirable to minimize these fill-in terms, so that both computational time/effort and computer memory requirements can be substantially reduced. For example, the following matrix $[A]$ and vector $[b]$ are given:

$$[A] = \begin{bmatrix} 112 & 7 & 0 & 0 & 0 & 2 \\ 7 & 110 & 5 & 4 & 3 & 0 \\ 0 & 5 & 88 & 0 & 0 & 1 \\ 0 & 4 & 0 & 66 & 0 & 0 \\ 0 & 3 & 0 & 0 & 44 & 0 \\ 2 & 0 & 1 & 0 & 0 & 11 \end{bmatrix} \quad (33)$$

$$[b] = \begin{bmatrix} 121 \\ 129 \\ 94 \\ 70 \\ 47 \\ 14 \end{bmatrix} \quad (34)$$

The Cholesky factorization matrix $[U]$, based on the original matrix $[A]$ (see Equation 33) and Equations (6-7), or Figure 1, can be symbolically computed as:

$$[U] = \begin{bmatrix} \times & \times & 0 & 0 & 0 & \times \\ 0 & \times & \times & \times & \times & F \\ 0 & 0 & \times & F & F & \times \\ 0 & 0 & 0 & \times & F & F \\ 0 & 0 & 0 & 0 & \times & F \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \quad (35)$$

In Equation (35), the symbols \times , and F represents the “non-zero” and “fill-in” terms, respectively.

In practical applications, however, it is always a necessary step to rearrange the original matrix $[A]$ through re-ordering algorithms (or subroutines) [Refs 1-2] and produce the following integer mapping array

$$IPERM (\text{new equation \#}) = \{\text{old equation \#}\} \quad (36)$$

such as, for this particular example:

$$IPERM \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad (37)$$

Using the above results (see Equation 37), one will be able to construct the following re-arranged matrices:

$$[A^*] = \begin{bmatrix} 11 & 0 & 0 & 1 & 0 & 2 \\ 0 & 44 & 0 & 0 & 3 & 0 \\ 0 & 0 & 66 & 0 & 4 & 0 \\ 1 & 0 & 0 & 88 & 5 & 0 \\ 0 & 3 & 4 & 5 & 110 & 7 \\ 2 & 0 & 0 & 0 & 7 & 112 \end{bmatrix} \quad (38)$$

and

$$[b^*] = \begin{bmatrix} 14 \\ 47 \\ 70 \\ 94 \\ 129 \\ 121 \end{bmatrix} \quad (39)$$

Remarks:

- In the original matrix A (shown in Equation 33), the nonzero term A (old row 1, old column 2) = 7 will move to new location of the new matrix A^* (new row 6, new column 5) = 7, etc.
- The non zero term A (old row 3, old column 3) = 88 will move to A^* (new row 4, new column 4) = 88, etc.
- The value of b (old row 4) = 70 will be moved to (or located at) b^* (new row 3) = 70, etc.

Now, one would like to solve the following modified system of linear equations (SLE) for $[x^*]$,

$$[A^*][x^*] = [b^*] \quad (40)$$

rather than to solve the original SLE (see Equation1). The original unknown vector $\{x\}$ can be easily recovered from $[x^*]$ and $[IPERM]$, shown in Equation (37).

The factorized matrix $[U^*]$ can be “symbolically” computed from $[A^*]$ as (by referring to either Figure 1, or Equations 6-7):

$$[U^*] = \begin{bmatrix} \times & 0 & 0 & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & \times & 0 \\ 0 & 0 & 0 & \times & \times & F \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix} \quad (41)$$

You can clearly see the big benefits of solving the SLE shown in Equation (40), instead of solving the original Equation (1), since the factorized matrix $[U^*]$ has only 1 fill-in term (see the symbol “ F ” in Equation 41), as compared to six fill-in-terms occurred in the factorized matrix $[U]$ (shown in Equation 35)!

On-Line Chess-Like Game For Reordering/Factorized Phase [4].

Based on the discussions presented in the previous section 2 (about factorization phase), and section 3 (about reordering phase), one can easily see the similar operations between the symbolic, numerical factorization and reordering (to minimize the number of fill-in terms) phases of sparse SLE.

In practical computer implementation for the solution of SLE, the reordering phase is usually conducted first (to produce the mapping between “old↔new” equation numbers, as indicated in the integer array IPERM(-), see Equations 36-37).

Then, the sparse “symbolic” factorization phase is followed by using either Cholesky Equations 6-7, or the LDL^T Equations 24-25 (without requiring the actual/numerical values to be computed). The reason is because during the “symbolic factorization” phase, one only wishes to find the number (and the location) of non-zero “fill-in terms”. This “symbolic” factorization process is necessary for allocating the “computer memory” requirement for the “numerical factorization” phase which will actually compute the exact numerical values of $[U^*]$, based on the same Cholesky Equations (6-7) (or the LDL^T Equations (24-25)).

In this work, a chess-like game (shown in Figure 2, Ref. [4]) has been designed with the following objectives:

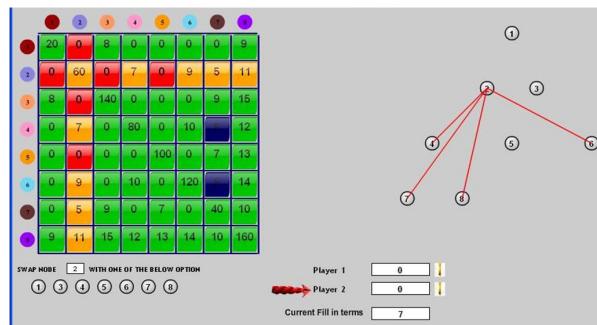


Figure 2 A Chess-Like Game For Learning to Solve SLE.

(A)Teaching undergraduate/HS students the process how to use the reordering output IPERM(-), see Equations (36-37) for converting the original/given matrix $[A]$, see Equation (33), into the new/modified matrix $[A^*]$, see Equation (38). This step is reflected in Figure 2, when the “Game Player” decides to swap node (or equation) “ i ” (say $i = 2$) with another node (or equation) “ j ”, and click the “CONFIRM” icon!

Since node “ $i = 2$ ” is currently connected to nodes $j = 4,6,7,8$; hence swapping node $i = 2$ with the above nodes “ j ” will “NOT” change the number/pattern of “Fill-in” terms. However, if node $i = 2$ is swapped with node $j = 1, or 3, or 5$, then the fill-in terms pattern may change (for better or worse)!

(B) Helping undergraduate/HS students to understand the “symbolic” factorization” phase, by symbolically utilizing the Cholesky factorized Equations (6-7). This step is illustrated in Figure 2, for which the “game player” will see (and also hear the computer animated sound, and human

voice), the non-zero terms (including fill-in terms) of the original matrix $[A]$ to move to the new locations in the new/modified matrix $[A^*]$.

(C) Helping undergraduate/HS students to understand the “numerical factorization” phase, by numerically utilizing the same Cholesky factorized Equations (6-7).

(D) Teaching undergraduate engineering/science students and even high-school (HS) students to “understand existing reordering concepts”, or even to “discover new reordering algorithms”

Further Explanation On The Developed Game

1. In the above Chess-Like Game, which is available on-line [4], powerful features of FLASH computer environments [3], such as animated sound, human voice, motions, graphical colors etc... have all been incorporated and programmed into the developed game-software for more appealing to game players/learners.

2. In the developed “Chess-Like Game”, fictitious monetary (or any kind of ‘scoring system’) is rewarded (and broadcasted by computer animated human voice) to game players, based on how he/she swaps the node (or equation) numbers, and consequently based on how many fill-in “ F ” terms occurred. In general, less fill-in terms introduced will result in more rewards!

3. Based on the original/given matrix $[A]$, and existing re-ordering algorithms (such as the Reverse Cuthill-McKee, or RCM algorithms [1-2]) the number of fill-in (“ F ”) terms can be computed (using RCM algorithms). This internally generated information will be used to judge how good the players/learners are, and/or broadcast “congratulations message” to a particular player who discovers new “chess-like move” (or, swapping node) strategies which are even better than RCM algorithms!

4. Initially, the player(s) will select the matrix size (8×8 , or larger is recommended), and the percentage (50%, or larger is suggested) of zero-terms (or sparsity of the matrix). Then, “START Game” icon will be clicked by the player.

5. The player will then CLICK one of the selected node “ i ” (or equation) numbers appearing on the computer screen. The player will see those nodes “ j ” which are connected to node “ i ” (based on the given/generated matrix $[A]$). The player then has to decide to swap node “ i ” with one of the possible node “ j ”. After confirming the player’s decision, the outcomes/results will be announced by the computer animated human voice, and the monetary-award will (or will NOT) be given to the players/learners, accordingly. In this software, a maximum of \$1,000,000 can be earned by the player, and the “exact dollar amount” will be INVERSELY proportional to the number of fill-in terms occurred (as a consequence of the player’s decision on how to swap node “ i ” with another node “ j ”).

6. The next player will continue to play, with his/her move (meaning to swap the i^{th} node with the j^{th} node) based on the current best non-zero terms pattern of the matrix.

Multiple Choice Tests

Cholesky Algorithms

For a given SLE:

$$[A][x] = [b]$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0.5 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0.5 & 0 & -1 & 1 \end{bmatrix}$$

and

$$[b] = \begin{bmatrix} -0.5 \\ 5 \\ -5 \\ 1.5 \end{bmatrix}$$

1. The Cholesky factorized matrix $[U]$ can be computed as

$$(a) \begin{bmatrix} 1.414 & 0.7071 & 0 & 0.3536 \\ 0 & 1.225 & -0.8165 & 0.2041 \\ 0 & 0 & 1.155 & -0.7217 \\ 0 & 0 & 0 & 0.5590 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1.414 & -0.7071 & 0 & 0.3536 \\ 0 & 1.225 & -0.8165 & 0.2041 \\ 0 & 0 & 1.155 & -0.7217 \\ 0 & 0 & 0 & 0.5590 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1.414 & 0.7071 & 0 & -0.3536 \\ 0 & 1.225 & -0.8165 & 0.2041 \\ 0 & 0 & 1.155 & -0.7217 \\ 0 & 0 & 0 & 0.5590 \end{bmatrix}$$

$$(d) \begin{bmatrix} 1.414 & 0.7071 & 0 & 0.3536 \\ 0 & 1.225 & -0.8165 & 0.2041 \\ 0 & 0 & -1.155 & -0.7217 \\ 0 & 0 & 0 & 0.5590 \end{bmatrix}$$

2. The forward solution vector $[y]$ can be computed as

- (a) $\vec{y}^T = \{0.5363, 38.784, -15.877, 0.5590\}$
- (b) $\vec{y}^T = \{0.5363, -15.877, 38.784, 0.5590\}$
- (c) $\vec{y}^T = \{-3.536, -1.5877, 3.878, 0.5590\}$
- (d) $\vec{y}^T = \{-0.3536, 3.8784, -1.5877, -0.5590\}$

3. The backward solution vector $[x]$ can be computed as

- (a) $\vec{x}^T = \{1, 2, -2, -1\}$
- (b) $\vec{x}^T = \{1, 2, 2, -1\}$
- (c) $\vec{x}^T = \{-1, 2, -2, 1\}$
- (d) $\vec{x}^T = \{1, 2, 2, 1\}$

4. The determinant of $[A]_{4 \times 4}$ can be computed as:

- (a) -5
- (b) 5
- (c) -50
- (d) 1.25

5. Based on the given matrix $[A]$, and assuming the reordering algorithm will produce the

following mapping IPERM (new equation #) = {old equation #}, such as $IPERM \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$. The

non zero off-diagonal term $A(\underline{\text{old}} \text{ row 4}, \underline{\text{old}} \text{ column 1}) = 0.5$ will move to the following new location of the new matrix $[A^*]$

- (a) $A^* (\underline{\text{new}} \text{ row 3}, \underline{\text{new}} \text{ column 1})$
- (b) $A^* (\underline{\text{new}} \text{ row 1}, \underline{\text{new}} \text{ column 3})$
- (c) $A^* (\underline{\text{new}} \text{ row 3}, \underline{\text{new}} \text{ column 2})$

(d) A^* (new row 2, new column 2)

6. Based on the given matrix $[A]$, and the given reordering mapping $IPERM \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$, the non-zero diagonal term $A(4,4) = 1$ will move to the following new location of the new matrix $[A^*]$

- (a) $A^*(1,1) = 1$
- (b) $A^*(2,2) = 1$
- (c) $A^*(3,3) = 1$
- (d) $A^*(4,4) = 1$

LDL^T Algorithms

For the given SLE:

$$[A][x] = [b]$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0.5 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0.5 & 0 & -1 & 1 \end{bmatrix}$$

and

$$[b] = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0.5 \end{bmatrix}$$

7. The lower triangular matrix $[L]$ can be computed as

$$(a) [L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & -0.6667 & 1 & 0 \\ 0.25 & 0.1667 & -0.625 & 1 \end{bmatrix}$$

$$(b) [L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0.6667 & 1 & 0 \\ 0.25 & 0.1667 & -0.625 & 1 \end{bmatrix}$$

$$(c) [L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & -0.6667 & 1 & 0 \\ 0.25 & 0.1667 & 0.625 & 1 \end{bmatrix}$$

$$(d) [L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & -0.6667 & 1 & 0 \\ 0.25 & 0.1667 & -0.625 & 1 \end{bmatrix}$$

8. The diagonal matrix $[D]$ can be computed as:

$$(a) [D] = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.3333 & 0 \\ 0 & 0 & 0 & 0.3125 \end{bmatrix}$$

$$(b) [D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 \\ 0 & 0 & 1.3333 & 0 \\ 0 & 0 & 0 & 0.3125 \end{bmatrix}$$

$$(c) [D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.3333 & 0 \\ 0 & 0 & 0 & -0.3125 \end{bmatrix}$$

$$(d) [D] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.3333 & 0 \\ 0 & 0 & 0 & 0.3125 \end{bmatrix}$$

9. The forward solution for the unknown vector $[z]$, in $[L][z]=[b]$ can be computed as:

- (a) $\{z\}^T = \{-2, 0, 1, 0.625\}$
- (b) $\{z\}^T = \{2, 0, 1, 0.625\}$
- (c) $\{z\}^T = \{2, 0, -1, 0.625\}$
- (d) $\{z\}^T = \{2, 0, 1, -0.625\}$

10. The diagonal scaling solution for the unknown vector $[y]$ in $[D][y]=[z]$ can be computed as:

- (a) $\{y\}^T = \{-1, 0, 0.75, 2\}$
- (b) $\{y\}^T = \{1, 0, -0.75, 2\}$
- (c) $\{y\}^T = \{1, 0, 0.75, -2\}$
- (d) $\{y\}^T = \{1, 0, 0.75, 2\}$

11. The backward solution for the original unknown vector $[x]$, in $[L]^T[x]=[y]$, can be computed as:

- (a) $\{x\}^T = \{1, 1, 2, 2\}$
- (b) $\{x\}^T = \{2, 1, 2, 1\}$
- (c) $\{x\}^T = \{1, 1, 2, 1\}$
- (d) $\{x\}^T = \{2, 2, 2, 1\}$

12. Given the following 6x6 matrix $[A]$, which is assumed to be SPD:

$$[A] = \begin{bmatrix} \times & 0 & \times & 0 & \times & 0 \\ & \times & 0 & \times & 0 & 0 \\ & & \times & 0 & \times & \times \\ \text{Sym} & & & \times & 0 & 0 \\ & & & & \times & 0 \\ & & & & & \times \end{bmatrix}$$

where \times = a nonzero value (given)

0 = a zero value (given)

Based on the numerically factorized formulas (shown in Equations 6-7), or even more helpful information as indicated in Figure 1, the symbolically factorized (upper-triangular) matrix can be obtained as:

$$(a) [U] = \begin{bmatrix} * & 0 & * & 0 & * & 0 \\ & * & 0 & * & 0 & 0 \\ & & * & 0 & * & * \\ & & & * & 0 & 0 \\ & & & & * & F \\ & & & & & * \end{bmatrix}$$

where

* = a nonzero value (computed, at the same location as the original nonzero value of $[A]$)

0 = a zero value

F = a nonzero Fill-in-Term (computed)

Note

$$U(5,6) = F$$

$$A(5,6) = 0$$

$$(b) [U] = \begin{bmatrix} * & 0 & * & 0 & * & 0 \\ & * & 0 & F & 0 & 0 \\ & & * & 0 & * & * \\ & & & * & 0 & 0 \\ & & & & * & F \\ & & & & & * \end{bmatrix}$$

$$(c) [U] = \begin{bmatrix} * & 0 & * & 0 & * & 0 \\ * & 0 & F & 0 & 0 & \\ * & 0 & F & * & & \\ * & 0 & 0 & & & \\ * & F & & & & \\ * & & & & & \end{bmatrix}$$

$$(d) [U] = \begin{bmatrix} * & 0 & * & 0 & * & 0 \\ * & 0 & F & 0 & 0 & \\ * & 0 & F & F & & \\ * & 0 & 0 & & & \\ * & F & & & & \\ * & & & & & \end{bmatrix}$$

References

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CHOLESKY AND LDL^T DECOMPOSITION

Topic Cholesky and LDL^T Decomposition

Summary Textbook notes on Cholesky and LDL^T Decomposition

Major General Engineering

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