Chapter 07.08
Simpson $\frac{3}{8}$ Rule For Integration.

Introduction

The main objective in this chapter is to develop appropriated formulas for obtaining the integral expressed in the following form:

$$I = \int_a^b f(x)dx \tag{1}$$

where $f(x)$ is a given function. Most (if not all) of the developed formulas for integration is based on a simple concept of replacing a given (oftently complicated) function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$, where i represents the order of the polynomial function. In the previous chapter, it has been explained and illustrated that Simpsons $\frac{1}{3}$ rule for integration can be derived by replacing the given function $f(x)$ with the 2nd –order (or quadratic) polynomial function $f_i(x) = f_2(x)$, defined as:
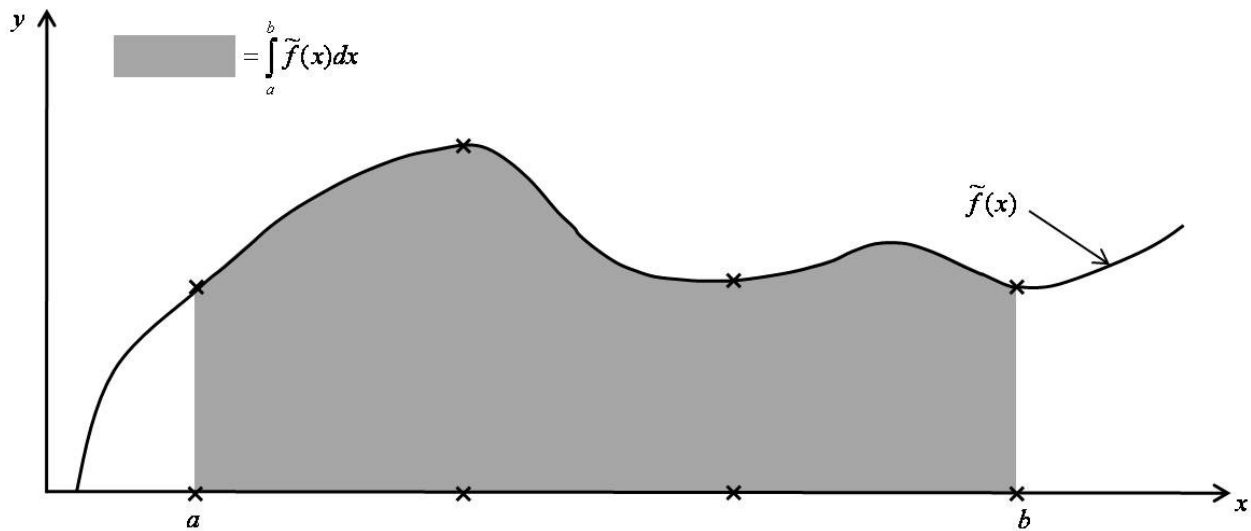
$$f_2(x) = a_0 + a_1 x + a_2 x^2 \tag{2}$$

Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson $\frac{3}{8}$ rule for integration can be derived by replacing the given function $f(x)$ with the 3rd-order (or cubic) polynomial (passing through 4 known data points) function $f_i(x) = f_3(x)$, defined as

$$f_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \Bigg\} \tag{3}$$

which can also be symbolically represented in Figure 1.

<u>Method 1</u>

The unknown coefficients a_0, a_1, a_2 and a_3 (in Equation (3)) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}\}$ into Equation (3), as following

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2 \end{aligned}\right\} \tag{4}$$

Equation (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \tag{5}$$

The above Equation (5) can be symbolically represented as
$$[A]_{4\times4}\, \vec{a}_{4\times1} = \vec{f}_{4\times1} \tag{6}$$

Thus,

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A]^{-1} \times \vec{f} \tag{7}$$

Substituting Equation (7) into Equation (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \tag{8}$$

<u>Remarks</u>

(a) As indicated in Figure 1, one has

$$
\left.\begin{aligned}
x_0 &= a \\
x_1 &= a + h \\
&= a + \frac{b-a}{3} \\
&= \frac{2a+b}{3} \\
x_2 &= a + 2h \\
&= a + \frac{2b-2a}{3} \\
&= \frac{a+2b}{3} \\
x_3 &= a + 3h \\
&= a + \frac{3b-3a}{3} \\
&= b
\end{aligned}\right\}
\tag{9}
$$

(b) With the help from MATLAB [Ref. 2], the unknown vector $\vec{a}$ (shown in Equation 7) can be solved as

Method 2

Using Lagrange interpolation, the cubic polynomial function $f_{i=3}(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$
\begin{aligned}
f_3(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_3) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)
\end{aligned}
\tag{10}
$$

Simpsons $\frac{3}{8}$ Rule For Integration

Thus, Equation (1) can be calculated as (See Equations 8, 10 for Method 1 and Method 2, respectively):

$$
I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx
$$

$$
I = (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8}
\tag{11}
$$

Since $h = \frac{b-a}{3}$

Hence $b-a = 3h$, and the above equation becomes:

$$I \approx \frac{3h}{8} \times \{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\} \tag{12}$$

The error introduced by the Simpson $\frac{3}{8}$ rule can be derived as [Ref. 1]:

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta), \text{ where } a \leq \zeta \leq b \tag{13}$$

<u>Example 1</u> (Single Simpson $\frac{3}{8}$ rule)

Compute $I = \int\limits_{a=8}^{b=30} \left\{ 2000\ln\left(\frac{140,000}{140,000 - 2100x}\right) - 9.8x \right\} dx,$ by using a single segment Simpson $\frac{3}{8}$ rule

<u>Solution</u>
In this example:

$$h = \frac{b-a}{3}$$
$$= \frac{30-8}{3}$$
$$= 7.3333$$

$$x_0 = 8 \Rightarrow f(x_0) = 2000\ln\left(\frac{140000}{140000 - 2100 \times 8}\right) - 9.8 \times 8$$
$$= 177.2667$$

$$\begin{cases} x_1 = x_0 + h \\ \quad = 8 + 7.3333 \\ \quad = 15.3333 \\ f(x_1) = 2000\ln\left(\frac{140000}{140000 - 2100 \times 15.3333}\right) - 9.8 \times 15.3333 \\ \quad = 372.4629 \end{cases}$$

$$
\begin{cases}
x_2 = x_0 + 2h \\
\quad = 8 + 2(7.3333) \\
\quad = 22.6666 \\
f(x_2) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 \\
\quad = 608.8976
\end{cases}
$$

$$
\begin{cases}
x_3 = x_0 + 3h \\
\quad = 8 + 3(7.3333) \\
\quad = 30 \\
f(x_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 \\
\quad = 901.6740
\end{cases}
$$

Applying Equation (12), one has:

$$
I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}
$$

$$
I = 11063.3104
$$

The "exact" answer can be computed as
$$
I_{exact} = 11061.34
$$

Multiple Segments for Simpson $\dfrac{3}{8}$ Rule

Using $"n" =$ number of equal (small) segments, the width $"h"$ can be defined as

$$
h = \frac{b-a}{3} \tag{14}
$$

Notes:
n = multiple of 3 = number of small $"h"$ segments
The integral, shown in Equation (1), can be expressed as

$$
I = \int_a^b f(x)\,dx \approx \int_a^b f_3(x)\,dx
$$

$$
I \approx \int_{x_0=a}^{x_3} f_3(x)\,dx + \int_{x_3}^{x_6} f_3(x)\,dx + \dots\dots + \int_{x_{n-3}}^{x_n=b} f_3(x)\,dx \tag{15}
$$

Substituting Simpson $\tfrac{3}{8}$ rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{\begin{array}{l} f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6) \\ +.....+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n) \end{array}\right\} \tag{16}$$

$$I = \frac{3h}{8}\left\{ f(x_0)+3\sum_{i=1,4,7,...}^{n-2} f(x_i)+3\sum_{i=2,5,8,...}^{n-1} f(x_i)+2\sum_{i=3,6,9,...}^{n-3} f(x_i)+f(x_n)\right\} \tag{17}$$

<u>Example 2</u> (Multiple segments Simpson $\tfrac{3}{8}$ rule)

Compute $I = \displaystyle\int_{a=8}^{b=30}\left\{2000\ln\left(\frac{140,000}{140,000-2100x}\right)-9.8x\right\}dx$, using Simple $\tfrac{3}{8}$ multiple segments rule,

with number (of "h") segments $= n = 6$ (which corresponds to 2 "big" segments).

<u>Solution</u>
In this example, one has (see Equation 14):
$$h = \frac{30-8}{6} = 3.6666$$
$\{x_0, f(x_0)\} = \{8, 177.2667\}$
$\{x_1, f(x_1)\} = \{11.6666, 270.4104\};$ *where* $x_1 = x_0 + h = 8 + 3.6666 = 11.6666$
$\{x_2, f(x_2)\} = \{15.3333, 372.4629\};$ *where* $x_2 = x_0 + 2h = 15.3333$
$\{x_3, f(x_3)\} = \{19, 484.7455\};$ *where* $x_3 = x_0 + 3h = 19$
$\{x_4, f(x_4)\} = \{22.6666, 608.8976\};$ *where* $x_4 = x_0 + 4h = 22.6666$
$\{x_5, f(x_5)\} = \{26.3333, 746.9870\};$ *where* $x_5 = x_0 + 5h = 26.3333$
$\{x_6, f(x_6)\} = \{30, 901.6740\};$ *where* $x_6 = x_0 + 6h = 30$

Applying Equation (17), one obtains:

$$I = \frac{3}{8}(3.6666)\left\{177.2667+3\sum_{i=1,4,...}^{n-2=4} f(x_i)+3\sum_{i=2,5,...}^{n-1=5} f(x_i)+2\sum_{i=3,6,...}^{n-3=3} f(x_i)+901.6740\right\}$$

$$I = (1.3750)\{177.2667+3(270.4104+608.8976)+3(372.4629+746.9870)+2(484.7455)+901.6740\}$$

$$I = 11,601.4696$$

Example 3 (Mixed, multiple segments Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rules)

Compute $I = \int\limits_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x}\right) - 9.8x \right\} dx$, using Simpson $\frac{1}{3}$ rule (with $n_1 = 4$ small

segments), and Simpson $\frac{3}{8}$ rule (with $n_2 = 3$ small segments).

Solution:
In this example, one has:

$$h = \frac{b-a}{n}$$
$$= \frac{b-a}{n_1 + n_2}$$
$$= \frac{30-8}{(4+3)}$$
$$= 3.1429$$

$$\left.\begin{aligned}
x_0 &= a = 8 \\
x_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\
x_2 &= x_0 + 2h = 8 + 2(3.1429) = 14.2857 \\
x_3 &= x_0 + 3h = 8 + 3(3.1429) = 17.4286 \\
x_4 &= x_0 + 4h = 8 + 4(3.1429) = 20.5714
\end{aligned}\right\} Simpson\ \frac{1}{3}\ rule$$

$$x_5 = x_0 + 5h = 8 + 5(3.1429) = 23.7143$$
$$x_6 = x_0 + 6h = 8 + 6(3.1429) = 26.8571$$
$$x_7 = x_0 + 7h = 8 + 7(3.1429) = 30$$

$$f(x_0 = 8) = 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100 \times 8}\right) - 9.8 \times 8 = 177.2667$$

Similarly:

$$f(x_1 = 11.1429) = 256.5863$$
$$f(x_2) = 342.3241$$
$$f(x_3) = 435.2749$$
$$f(x_4) = 536.3909$$
$$f(x_5) = 646.8260$$
$$f(x_6) = 767.9978$$
$$f(x_7) = 901.6740$$

For multiple segments $(n_1 = first\ 4\ segments)$, using Simpson $\frac{1}{3}$ rule, one obtains (See Equation 19):

$$I_1 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3}f(x_i) + 2\sum_{i=2,\ldots}^{n_1-2=2}f(x_i) + f(x_{n_1})\right\}$$

$$I_1 = \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$I_1 = 4364.1197$$

For multiple segments $(n_2 = last\ 3\ segments)$, using Simpson $\frac{3}{8}$ rule, one obtains (See Equation 17):

$$I_2 = \left(\frac{3h}{8}\right)\left\{f(x_0) + 3\sum_{i=1,3,\ldots}^{n_2-2=1}f(x_i) + 3\sum_{i=2,\ldots}^{n_2-1=2}f(x_i) + 2\sum_{i=3,6,\ldots}^{n_2-3=0}f(x_i) + f(x_{n_1})\right\}$$

$$I_2 = \left(\frac{3}{8}\times3.1429\right)\{177.2667 + 3(256.5863) + 3(342.3241) + (skip!) + 435.2749\}$$

$$I_2 = 6697.2748$$

The mixed (combined) Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rules give:

$$I = I_1 + I_2 = 4364.1197 + 6697.2748$$

$$I = 11,061.3946$$

Remarks:

(a) Comparing the truncated error of Simpson $\frac{1}{3}$ rule

$$E_t = -\frac{(b-a)^5}{2880}\times f''''(\zeta) \tag{18}$$

With Simple $\frac{3}{8}$ rule (See Equation 12), the latter seems to offer <u>slightly more accurate</u> answer than the former. However, the cost associated with Simpson $\frac{3}{8}$ rule (using 3rd order polynomial function) is significant higher than the one associated with Simpson $\frac{1}{3}$ rule (using 2nd order polynomial function).

(b) The number of multiple segments that can be used in the conjunction with Simpson $\frac{1}{3}$ rule is 2,4,6,8,.. (any even numbers).

$$I_1 = \left(\frac{h}{3}\right)\{f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \ldots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$

$$I_2 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_i) + f(x_n)\right\} \tag{19}$$

However, Simpson $\frac{3}{8}$ rule can be used with the number of segments equal to 3,6,9,12,.. (can be either certain odd or even numbers).

(c) If the user wishes to use, say 7 segments, then the mixed Simpson $\frac{1}{3}$ rule (for the first 4 segments), and Simpson $\frac{3}{8}$ rule (for the last 3 segments).

Computer Algorithm For Mixed Simpson $\frac{1}{3}$ and $\frac{3}{8}$ Rule For Integration

Based on the earlier discussions on (Single and Multiple segments) Simpson $\frac{1}{3}$ and $\frac{3}{8}$ rules, the following "pseudo" step-by-step mixed Simpson rules can be given as

Step 1 User's input information, such as

Given function $f(x)$, integral limits *"a,b"*,
n_1 = number of small, "h" segments, in conjunction with Simpson $\frac{1}{3}$ rule.
n_2 = number of small, "h" segments, in conjunction with Simpson $\frac{3}{8}$ rule.

Notes:
n_1 = a multiple of 2 (any <u>even</u> numbers)
n_2 = a multiple of 3 (can be certain <u>odd</u>, or <u>even</u> numbers)

Step 2
Compute $n = n_1 + n_2$
$$h = \frac{b-a}{n}$$

$x_0 = a$

$x_1 = a + 1h$

$x_2 = a + 2h$

.

.

$x_i = a + ih$

.

.

$x_n = a + nh = b$

Step 3

Compute "multiple segments" Simpson $\frac{1}{3}$ rule (See Equation 19)

$$I_1 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,...}^{n_1-1}f(x_i) + 2\sum_{i=2,4,6...}^{n_1-2}f(x_i) + f(x_n)\right\}$$
(19, repeated)

Step 4

Compute "multiple segments" Simpson $\frac{3}{8}$ rule (See Equation 17)

$$I_2 = \left(\frac{3h}{8}\right)\left\{f(x_0) + 3\sum_{i=1,4,7...}^{n_2-2}f(x_i) + 3\sum_{i=2,5,8...}^{n_2-1}f(x_i) + 2\sum_{i=3,6,9,...}^{n_2-3}f(x_i) + f(x_{n_2})\right\}$$
(17, repeated)

Step 5

$I = I_1 + I_2$
(20)

and print out the final approximated answer for I.

Multiple Choice Tests

1. Simpson $\frac{3}{8}$ rule for integration is mainly based upon the idea of

(a) Replacing $f(x)$, in $I = \int_a^b f(x)dx$, by a cubic (3rd order) polynomial function

(b) Replacing $f(x)$, in $I = \int_a^b f(x)dx$, by a quadratic (2nd order) polynomial function.

(c) Converting the limit of integral limits $[a,b]$ into $[-1,+1]$.

(d) Using similar concepts as Gauss Quadrature formula.

2. Given $f(x) = e^{-2x} + 4x^2 - 8$. The "exact" (analytical, closed form) solution for $I = \int_1^4 f(x)dx$, is

(a) 6.0067
(b) 5.7606
(c) 60.0675
(d) 67.6075

3. Given $f(x) = e^{-2x} + 4x^2 - 8$, using (single segment, n=3) Simpson $\frac{3}{8}$ rule, $I = \int_1^4 f(x)dx$ can be computed as.

(a)61.3740
(b) 60.0743
(c) 59.3470
(d) 58.8992

4. Given $f(x) = e^{-2x} + 4x^2 - 8$, using multiple segments $n = 6$ Simpson $\frac{3}{8}$ rule, $I = \int_1^4 f(x)dx$ can be computed as

(a) 60.8206
(b) 60.6028
(c) 60.0682
(d) 60.0675

5. Given $f(x) = e^{-2x} + 4x^2 - 8$, using mixed multiple segments Simpson $\frac{1}{3}$ rule (with $n_1 = 4$ segments) and Simpson $\frac{3}{8}$ rule (with $n_2 = 3$ segments), $I = \int_1^4 f(x)dx$ can be computed as.

(a) 60.0677
(b) 59.0677
(c) 61.0677
(d) 59.7607

6. Comparing Simpson $\frac{3}{8}$ rule truncated error formula (See Equation 13) with Simpson $\frac{1}{3}$ rule truncated error formula (See Chapter 07.03)

$E_t = -\dfrac{(b-a)^5}{2880} \times f''''(\zeta)$, where $a \le \zeta \le b$, the following conclusion can be made:

(a)Simpson $\frac{3}{8}$ rule is <u>significantly more accurate</u> than Simpson $\frac{1}{3}$ rule

(b) It is "worthy" in terms of computational efforts versus accuracy to use Simpson $\frac{3}{8}$ rule, instead of Simpson $\frac{1}{3}$ rule.

(c) It is "not worthy" in terms of computational efforts versus accuracy to use Simpson $\frac{3}{8}$ rule, instead of Simpson $\frac{1}{3}$ rule.

(d) Simpson $\frac{3}{8}$ rule is <u>Less Accurate</u> than Simpson $\frac{1}{3}$ rule.

Multiple Choice Answers

1. = (a)
2. = ©
3. = (b)
4. = ©
5. = (a)
6. = ©

References
[1] S.C. Chapra, R.P. Canale, Numerical Methods for Engineers, Fourth Edition, Mc-Graw Hill.
[2] MATLAB from MathWorks.

SIMPSON $\frac{3}{8}$ RULE FOR INTEGRATION	
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