Domain Decomposition With Simultaneous Symmetrizing and Preconditioning Formulation

$$[K]\vec{Z} = \vec{f} \tag{1}$$

unsymmetric (in values, and locations)
non-singular

$$\begin{bmatrix} K_{ii}^r & K_{ib}^r \\ K_{bi}^r & K_{bb}^r \end{bmatrix} \begin{Bmatrix} z_i^r \\ z_b^r \end{Bmatrix} = \begin{Bmatrix} f_i^{(r)} \\ f_b^{(r)} \end{Bmatrix} \tag{2}$$

unsym

unsym

From Eq. (2A) $\Rightarrow$ $$z_i^r = [K_{ii}^r]^{-1}\{f_i^r - K_{ib}^r z_b^r\} \tag{3}$$

From Eq. (2B) $\Rightarrow$ $$K_{bi}^r z_i^r + K_{bb}^r z_b^r = f_b^r \tag{4}$$

Substitute Eq. (3) into Eq. (4) to get:

$$K_{bi}^r\left([K_{ii}^r]^{-1}\{f_i^r - K_{ib}^r \underline{\underline{z_b^r}}\}\right) + K_{bb}^r \underline{\underline{z_b^r}} = f_b^r \tag{5}$$

or
$$\underbrace{\left(K_{bb}^r - K_{bi}^r [K_{ii}^r]^{-1} K_{ib}^r\right)}_{\overline{K}_B^{(r)}} z_b^r = \underbrace{f_b^r - K_{bi}^r [K_{ii}^r]^{-1} f_i^r}_{\overline{f}_B^{(r)}} \tag{6}$$

$$\overline{K}_B^{(r)} * \vec{z}_b^r = \overline{f}_B^{(r)} \tag{7}$$

$$\overline{K}_B = \sum_{r=1}^{NSU} \overline{K}_B^{(r)} = \sum_r K_{BB}^{(r)} - \sum_r K_{bi}^r [K_{ii}^r]^{-1} K_{ib}^r \tag{8}$$

$$\overline{f}_B = \sum_r \overline{f}_B^{(r)} = \sum_r f_b^r - \sum_r K_{bi}^r [K_{ii}^r]^{-1} f_i^r \tag{9}$$

unsym

unsym

$$[\overline{K}_B]\vec{Z}_B = \overline{f}_B \tag{10}$$

How do we solve "system" boundary unknowns Z_b from Eq. (10) ??

Option 1 : Call GMRES to solve Eq. (10), and use unsym. sparse direct solver to perform the operations $[K_{ii}^r]^{-1} * \vec{d}^1_{\cancel{known}}$ in Eqs. (8–9), etc...

Option 2 : Call GMRES to solve Eq. (10), and use PCG + symmetrical sparse direct solver to perform the operations $[K_{ii_s}^r]^{-1} * \vec{d}^1$ .... (20)

in Eqs. (8–9), etc...

where $K_{ii_s}^r = \frac{1}{2}\left(K_{ii}^r + K_{ii}^{r\,T}\right) = \text{sym.}$ ———— (21)

Option 3 : Call PCG to solve the modified Eq. (10), as

$$[\bar{K}_B]^T [\bar{K}_{B_s}]^{-1} \bar{K}_B \vec{Z}_B = [\bar{K}_B]^T [\bar{K}_{B_s}]^{-1} \bar{f}_B \quad \text{........ (22)}$$

unsym

sym

and use unsym. sparse direct solver to perform $[K_{ii}^r]^{-1} * \vec{d}^1$

unsym

Option 4 : Call PCG to solve Eq. (22), and use PCG again + symmetrical sparse direct solver to perform operations shown in Eqs. (20–21)

GMRES

GMRES

Detailed Implementations for Option 4

see page 396 of Duc Nguyen's book [Year published = 2006]

Table 6.1 Preconditioned Conjugate Gradient Algorithm for Solving $[A]\vec{x} = \vec{b}$

Step 1: Initialized $\vec{x_o} = \vec{0}$

Step 2: Residual vector $\vec{r_o} = \vec{b}$ (or $\vec{r_o} = \vec{b} - \overrightarrow{Ax_0}$ for "any" initial guess $\vec{x_o}$)

Step 3: "Inexpensive" preconditioned $\vec{z_0} = [B]^{-1} \cdot \vec{r_0}$

Step 4: Search direction $\vec{d_0} = \vec{z_0}$

For i = 0, 1, 2, …, maxiter

Step 5: $\alpha_i = \dfrac{r_i^T z_i}{d_i^T \{A \cdot d_i\}}$

Step 6: $x_{i+1} = x_i + \alpha_i d_i$

Step 7: $r_{i+1} = r_i - \alpha_i [Ad_i]$

Step 8: Convergence check: if $\|r_{i+1}\| < \|r_0\| \cdot \varepsilon \rightarrow$ stop

Step 9: $z_{i+1} = B^{-1} r_{i+1}$

Step 10: $\beta_i = \dfrac{r_{i+1}^T z_{i+1}}{r_i^T z_i}$

Step 11: $d_{i+1} = z_{i+1} + \beta_i d_i$

End for

Remarks About Table 6.1 :

(a) Major operations involve Sparse (or Dense) Matrix times a column vector shown in steps # 2, #5

(b) Extra/Additional "inexpensive" preconditioned matrix are done in Step # 3, #9

We may skip this operation, by selecting [B] = Identity Matrix

From Eq. (22), one can write

$$[A_{4.1}]_{SYM} \vec{Z}_B = \vec{b}^{4.1} \qquad (22^A)$$

where:

$$[A_{4.1}] \equiv [\bar{K}_B]^T [\bar{K}_{B_S}]^{-1} [\bar{K}_B] \qquad (22^B)$$

$$\vec{b}^{4.1} \equiv [\bar{K}_B]^T [\bar{K}_{B_S}]^{-1} \bar{f}_B \qquad (22^C)$$

Thus, $[A_{4.1}]$ & $\vec{b}^{4.1}$ plays same roles as $[A]$ & $\vec{b}$ in Table 6.1 !

and

$$\left[\bar{K}_{B_S}\right]_{SYM} = 0.5\left(\underbrace{\bar{K}_B + \bar{K}_B^T}_{unsym}\right) = \text{SYM. matrix} \qquad (22D)$$

How to compute $[A]\vec{d}_{\cancel{known}} = ??$, in Step # 5 of PCG Table 6.1

$$[A]\vec{d} = [A_4]\vec{d} \qquad (23)$$

$$= [\bar{K}_B]^T [\bar{K}_{B_S}]^{-1} [\bar{K}_B]\,\vec{d} \qquad (24)$$

→ d^1

→ d^2

→ d^3

where: $$d^1 = \bar{K}_B \vec{d} = \left(\sum_r \bar{K}_B^{(r)}\right)\vec{d} = \left(\sum_r K_{BB}^{(r)} - \sum_r K_{BI}^{(r)} \left[K_{II}^{(r)}\right]^{-1} K_{IB}^{(r)}\right)\vec{d} \qquad (25)$$

→ first

→ $\boxed{\text{second}}$

→ third

→ fifth

fourth

How to compute $\overrightarrow{second} \equiv \left[K_{ii}^{(r)}\right]^{-1} * \overrightarrow{first}$??

or $$\left[K_{ii}^{(r)}\right] * \overrightarrow{second} = \overrightarrow{first} \qquad (25)$$

unsym

Use SYM. Preconditioning on Eq. (25), one gets

$$\underbrace{\left[K_{ii}^{(r)}\right]^T \left[K_{ii_s}^{(r)}\right]^{-1} \left[K_{ii}^{(r)}\right]}\ \overrightarrow{second} = \underbrace{\left[K_{ii}^{(r)}\right]^T \left[K_{ii_s}^{(r)}\right]^{-1} \overrightarrow{first}} \qquad (26)$$

unsym SYM

$$\underset{SYM}{[A_{4.2}]} * \overrightarrow{second} = \vec{b}^{4.2} \qquad (27)$$

where: $$[A_{4.2}] = K_{ii}^{(r)^T} \left[K_{ii_s}^{(r)}\right]^{-1} K_{ii}^{(r)} \qquad (28)$$

$$\vec{b}^{4.2} = K_{ii}^{(r)^T} \left[K_{ii_s}^{(r)}\right]^{-1} * \overrightarrow{first} \qquad (29)$$

and $$K_{ii_s}^{(r)} = 0.5\left(K_{ii}^{r} + [K_{ii}^{r}]^T\right) \qquad (30)$$

Again, we can **use/call** **PCG** (see Table 6.1) to solve for the vector $\overrightarrow{second}$ in Eq. (27), where $[A_{4.2 \atop SYM}]$ and $\overrightarrow{b^{4.2}}$ play same roles as $[A]$ & $\vec{b}$ in Table 6.1. For this purpose, one needs to compute

$$[A]\vec{d}_{\cancel{known}} = [A_{4.2}]\vec{d} \qquad (31)$$

$$= \underbrace{K_{ii}^{(r)^T} \underbrace{[K_{ii_s}^{(r)}]^{-1} \underbrace{K_{ii}^{(r)} \vec{d}}_{\overrightarrow{\#1}}}_{\overrightarrow{\#2}}}_{\overrightarrow{\#3}} \qquad (32)$$

How to compute $\vec{d}^2 = [\overline{K}_{B_s}]^{-1} * \vec{d}^1$, as shown in Eq. (24) ??

or $[\overline{K}_{B_s}]\vec{d}^2 = \vec{d}^1 \qquad (33)$

$$\underbrace{0.5(\overline{K}_B + \overline{K}_B^T)}\vec{d}^2 = \vec{d}^1 \qquad (34)$$

both unsym.

or $[A_{4.3 \atop SYM}]\vec{d}^2 = \vec{d}^1$ — we can use PCG & direct (sym.) sparse solver to solve for vector $\vec{d}^2$ (35)

$\boxed{\text{Again, we do we compute } [A_{4.3 \atop SYM}] * \vec{d}_{\cancel{known}} = ??}$

Already explained in Eq.(24)!

$$[A_{4.3}]\vec{d} = \frac{1}{2}\left(\overline{K}_B + \overline{K}_B^{T}\right)\vec{d} \qquad (36)$$

will be explaining on the following page(s)....

$$[\overline{K}_B]^T \vec{d} = \left[\sum_r \overline{K}_B^{(r)}\right]^T \vec{d} = ?? \qquad (37)$$

$$\left[\sum_r \underbrace{K^r_{BB}}_{unsym} - \sum_r K^r_{BI}\underbrace{[K^r_{II}]^{-1}}_{unsym} K^r_{ib}\right]^T \vec{d} = ?? \qquad (38)$$

$$\left(\sum_r [K^{(r)}_{BB}]^T - \sum_r K^r_{BI}\left[K^{(r)T}_{ii}\right]^{-1} K^r_{IB}\right)\vec{d} = ?? \qquad (39)$$

$\overrightarrow{\#11}$ = easy!

$\overrightarrow{\#12}$ → Need further explaination

$\overrightarrow{\#13}$ = easy

$\overrightarrow{\#14}$ = easy

$\overrightarrow{\#15}$ = easy

$$\overrightarrow{\#12} = \left[K^{(r)T}_{ii}\right]^{-1} * \overrightarrow{\#11} \overset{or}{\Longleftrightarrow} \left[K^{(r)T}_{ii}\right] * \overrightarrow{\#12} = \overrightarrow{\#11} \qquad (40)$$

Applying "Symmetrize pre-condition idea" to Eq.(40), one gets:

$$\left[K^{(r)T}_{ii}\right]^T \left[K^{(r)T}_{ii_s}\right]^{-1}\left[K^{(r)T}_{ii}\right] * \overrightarrow{\#12} = \left[K^{(r)T}_{ii}\right]^T\left[K^{(r)T}_{ii_s}\right]^{-1} * \overrightarrow{\#11} \qquad (41)$$

or
$$\underbrace{\left[K^{(r)}_{ii}\right]}_{unsym}\underbrace{\left[K^{T(r)}_{ii_s}\right]^{-1}}_{SYM}\underbrace{\left[K^{(r)T}_{ii}\right]}_{unsym} * \overrightarrow{\#12} = \underbrace{\left[K^{(r)}_{ii}\right]\left[K^{(r)}_{ii_s}\right]^{-1} * \overrightarrow{\#11}} \qquad (42)$$

or
$$\underset{SYM}{[A_{4.4}]} * \overrightarrow{\#12} = \vec{b}^{4.4} \qquad (43)$$

Again, use PCG here!

where $[K_{ii_s}^{(r)}]$ shown in Eqs. (41-43) is defined as:

$$[K_{ii_s}^{(r)}] = \boxed{\frac{1}{2}\left(K_{ii}^{(r)} + [K_{ii}^{(r)}]^T\right)} \equiv [K_{ii_s}^{(r)}]^T \quad (44)$$

can be explicitly computed ☺

Thus, when PCG is used to solve for unknown vector $\overrightarrow{\#12}$ one needs to compute:

$$\underset{SYM}{[A_{4.4}]} * \underset{\cancel{known}}{\vec{d}} = ?? \quad (45)$$

$$\underset{unsym.}{[K_{ii}^{(r)}]}\,[K_{ii_s}^{(r)T}]^{-1}\,\underset{unsym.}{[K_{ii}^{(r)T}]} * \vec{d} = ?? \quad (46)$$

SYM. & can be computed explicitly! ☺ $= [K_{ii_s}^{(r)}]$

$\overrightarrow{\#21}$

$\overrightarrow{\#22}$

$\overrightarrow{\#23}$

Remarks: Duc has found (through MATLAB numerical examples) that $[A_{4.2}] \equiv K_{ii}^{(r)T}[K_{ii_s}^{(r)}]^{-1}K_{ii}^{(r)}$ (see Eqs. 31-32) is the same as $[A_{4.4}] \equiv K_{ii}^{(r)}[K_{ii_s}^{(r)T}]^{-1}[K_{ii}^{(r)}]^T$ (see Eq. 46) for many cases like: $\underset{unsym}{[K_{ii}]} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, or $[K_{ii}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 59 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; or $K_{ii} = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 5 & 18 \\ 4 & 4 & 12 \end{bmatrix}$ etc..