How to efficiently transform a generalized Eigen-Problem into a standard problem??

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I.Introduction.

The objective here is to efficiently transform the following generalized eigen-problem:

$K\phi = \lambda M\phi$ …………………………………………………………………………. (1)

into the standard eigen-problem, where

K = $N \times N$, symmetrical Positive Definite (SPD) "Stiffness" matrix

M = $N \times N$, Symmetrical matrix

Remarks

1. If [M] is a "consistent" mass matrix, then [M] is a SPD matrix [Ref 1, page 854].
2. If [M] is a "lumped" mass matrix, with $m_{ii} > 0$ then [M] is SPD.
3. If [M] is a "lumped" mass, with some zero diagonal elements, then we first need to remove the massless dof by using the "static condensation" procedure [Ref.1, page 862].

Derivations:

Let $\phi = \begin{Bmatrix} \phi_m \\ \phi_s \end{Bmatrix}$ ……………………………………………………………………... (2)

Where $\phi_m =$ displacement vector related to the "master" (or mass) dof.

$\phi_s =$ displacement vector related to the "slave" (or massless) dof.

Hence, equation (1) can be partitioned as:

$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \begin{Bmatrix} \phi_m \\ \phi_s \end{Bmatrix} = \lambda \begin{bmatrix} M_{mm} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \phi_m \\ \phi_s \end{Bmatrix}$ ………………………………………………...(3)

From the second portion of equation (3), one gets:

$K_{sm}\phi_m + K_{ss}\phi_s = 0$ ……………………………………………………………………(4)

which can be solved for:

$\phi_s = -[K_{ss}]^{-1}\{K_{sm}\phi_m\}$………………………………………………………………….. (5)

from the first portion of the equation (3), one gets:

$K_{mm}\phi_m + K_{ms}\phi_s = \lambda M_{mm}\phi_m$ ………………………………………………………….(6)

substitute eq(5) into eq(6), one has:

$K_{mm}\phi_m + K_{ms}\left(-K_{ss}^{-1}\{K_{sm}\phi_m\}\right) = \lambda M_{mm}\phi_m$ ………………………………………………(7)

or:

$$\left[K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}\right]\phi_m = \lambda M_{mm}\phi_m \quad \text{...(8)}$$

Let:

$$K_{mm}^{*} = \left[K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}\right] \quad \text{...(9)}$$

Then, Eq (8) becomes:

$$K_{mm}^{*}\phi_m = \lambda_{mm}M_{mm}\phi_m \quad \text{...(10)}$$

where both

K_{mm}^{*} *and* M_{mm} matrices are SPD.

II. How to convert a generalized Evalue problem in to a standard one??

2.1 Method 1:

$K\phi = \lambda M\phi$ ……………………………………………………………………..(11)

Pre multiply both sides of equation 11 by $[K]^{-1}$; to obtain:

$\phi = \lambda K^{-1} M\phi$ ……………………………………………………………………..(12)

or:

$\left(\frac{1}{\lambda}\right)\phi = K^{-1} M\phi$ ……………………………………………………………………..(13)

Let $\lambda^* = \frac{1}{\lambda}$ (or $\lambda = \frac{1}{\lambda^*}$) ………………………………………………………….(14)

And let $K^* = K^{-1} M$ ……………………………………………………………...(15)

Hence, Eq(13) becomes:

$\lambda^* \phi = K^* \phi$ (Standard Evalue problem) ……………………………………………….(16)

Remarks:

(a) K^*, in general, will be <u>unsymmetrical</u>, even though both K and M are symmetrical matrices!

(b) K^*, in general, will be <u>nearly dense</u>, even though both K and M are sparse matrices!

(c) The original evalues λ (in Eq(1)) is the reciprocal of the eigen values λ^* in the <u>standard</u> Evalue problem (in Eq 16).

2.2 Method2:

(Factorizing the positive definite mass matrix M)

If [M] is SPD, then:

Let $\left[M\right]=\left[U\right]^T\left[U\right]$ by using cholesky factorzation ……………………………………(17)

Then, Eq (1) becomes:

$K\phi=\lambda\left[U^TU\right]\phi$ ……………………………………………………………….. (18)

Pre-multiply both sides of Eq(18) by U^{-T}, one obtains:

$U^{-T}K\phi=\lambda U^{-T}[U^TU]\phi$ ………………………………………………………... (19)
or

$U^{-T}K\phi=\lambda U\phi$ ……………………………………………………………….. (20)

Define:
$\tilde{\phi}=U\phi\,;\ \phi=U^{-1}\tilde{\phi}$ ……………………………………………………………… (21)

Then, Eq. (20) becomes:

$U^{-T}K(U^{-1}\tilde{\phi})=\lambda\tilde{\phi}$ ………………………………………………………………(22)

Define:

$K^*=U^{-T}KU^{-1}$ (=Symmetrical, since K is symmetrical) ……………………………(23)

$(\Rightarrow K^{*T}=[U^{-1}]^T[K]^T[U^{-T}]^T=U^{-T}KU^{-1}=K^*)$

Hence, Eq (22) becomes:
$K^*\tilde{\phi}=\lambda\tilde{\phi}$ (Standard Evalue Problem) ………………………………………………(24)

Remarks:

(a) The eigen-values λ of the generalized Eigen-equation (1) is the same as the eigen-values λ of the standard eigen-equation (24).

(b) In many iterative methods, one often needs to compute the following matrix-vector operation:

$\vec{x}=[K^*]\vec{d}=??$ ……………………………………………………………..(25)

Where $\vec{d}$ is a <u>known</u> vector, with compatible dimension.
Substituting Eq.(23) into Eq.(25), one gets:

$$\vec{x} = U^{-T} K \underbrace{\underbrace{U^{-1} \vec{d}}_{y^1}}_{y^2} \quad \text{.......(26)}$$

<u>Step 1:</u>

Compute $\vec{y}^1 = U^{-1} \vec{d}$; or $U \vec{y}^1 = \vec{d}$ ....(27)

Since we have already factorized $[M] = U^T U$ (see Eq.17), hence $\vec{y}^1$ can be solved by backward solution phase!

<u>Step 2:</u>

Compute $\vec{y}^2 = [K] \vec{y}^1$ (=Simple Matrix*Vector Operations) ....(28)

<u>Step 3:</u>

Compute $\vec{x} = U^{-T} \vec{y}^2$; or $U^T \vec{x} = \vec{y}^2$ -- (29)

Hence, $\vec{x}$ can be solved by forward solution phase!

(c) We do <u>not</u> have to explicitly form the nearly dense matrix K^* (shown in Eq.24), since in practical coding, we only need to deal with U and K (see Eq. 26).

2.3 Method 3:

(Factorizing the SPD stiffness matrix K)

Let $K = U^T U$ …………………………………………………………………………(30)

Then Eq.(1) becomes:

$\left[U^T U\right]\phi = \lambda M\phi$ …………………………………………………………………..(31)

Pre-multiplying both sides of Eq. (31) by U^{-T}, one obtains:

$U^{-T}\left[U^T U\right]\phi = \lambda U^{-T} M\phi$ ……………………………………………………………….(32)

or $U\phi = \lambda U^{-T} M\phi$ …………………………………………………………………...(33)

Define:

$\tilde{\phi} = U\phi$; or $\phi = U^{-1}\tilde{\phi}$ ……………………………………………………………(34)

Hence, Eq.(33) becomes:

$\tilde{\phi} = \lambda U^{-T} M(\phi = U^{-1}\tilde{\phi})$ ……………………………………………………………….(35)

or:

$\frac{1}{\lambda}\tilde{\phi} = U^{-T} M U^{-1}\tilde{\phi}$ ………………………………………………………………….. (36)

Define:

$\lambda^* = \frac{1}{\lambda}; or \quad \lambda = \frac{1}{\lambda^*}$ ………………………………………………………………….(37)

$M^* = U^{-T} M U^{-1} = [M^*]^T$, since [M] = symmetrical. ……………………………….. (38)

Then, Eq.(36) becomes:

$\lambda^* \tilde{\phi} = M^* \tilde{\phi}$ (standard Evalue problem) ……………………………………………(39)

Remarks:

(a) Eigen values ($= \lambda^*$) of the <u>standard</u> problem (see Eq 39) and λ of the <u>generalized</u> problem (see Eq.1) are the reciprocal of each other!

(b) Computation of matrix * known vector, such as:

$$\vec{x} = [M^*]\vec{d} \quad \text{...} \quad (40)$$

can be done efficiently, by substituting Eq.(38) into Eq.(40):

$$\vec{x} = \left[U^{-T} M U^{-1}\right]\vec{d} \quad \text{...} \quad (41)$$

------- step 1 = backward solution phase (see Eq 27)
---------- step 2 = matrix * vector operation (similar to Eq. 28)
-------------- step 3 = forward solution phase (see Eq.29)

(c) Again, we do <u>not</u> need to explicitly form the nearly dense matrix $\left[M^*\right]$, since we only need to deal with matrices [M] and [U], as indicated in Eq.(41).

<u>In conclusion:</u>

The user has a freedom to select either Method 2 or Method 3.
The choice should be dictated by the following considerations:

1. Which matrix, [K] or [M] (in Eq.1) is SPD?? and hence can be efficiently factorized??
2. Do we want to calculate the few lowest (or highest) eigen values of the original, generalized eigen-problem??

III. Near Future Research works.

3.1 How to incorporate Domain Decomposition and Parallel processing into either General (or standard) eigen-problem??

3.2 How to improve initial guess for eigen values (by using Gergorin theorms ??), or the initial guess for eigen-vectors (by using only 1 inverse iteration, see subspace iteration of Ref.1 ??)

3.3 Can we use “similar tricks” to convert system of linear, indefinite equations into SPD linear equations (Ref.2, Pages 154-163) to transform the generalized (or standard) eigen problem into a different problem which can be solved easier, faster and more robust??

IV. References:

1. K.J. Bathe, Finite Element Procedures, Prentice-Hall (1996)
2. D.T Nguyen, Finite Element Methods: Parallel-Sparse Statics and Eigen-Solutions, Springer (2006).