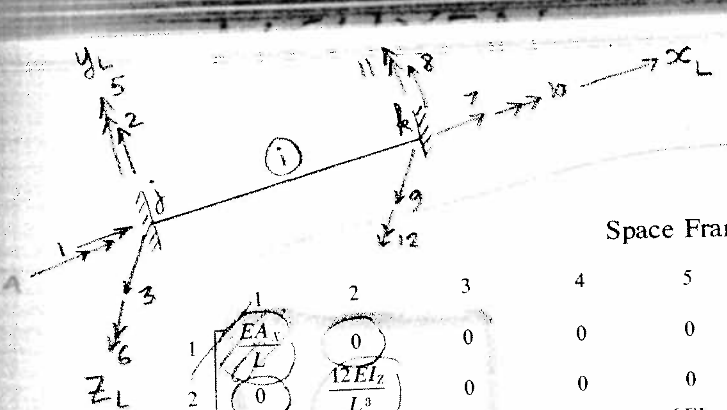
Element Stiffness Matrices & Local-Global Transformation for 2-D/3-D Truss, 2D/3-D Beam/Frame Elements



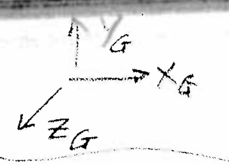
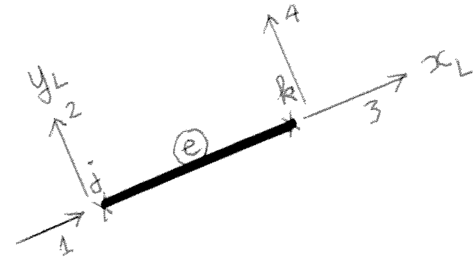


Table 4-1
Space Frame Member Stiffness Matrix

$$
\mathbf{S}_{Mi} =
\begin{array}{c}
\\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}\\
\left[\begin{array}{cccccc|cccccc}
\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} & 0 & -\frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2}\\
0 & 0 & \frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0 & 0 & 0 & -\frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0\\
0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0\\
0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0\\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L}\\
\hline
-\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2}\\
0 & 0 & -\frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 & 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0\\
0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0\\
0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0\\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L}
\end{array}\right]
\end{array}
$$



2-D Truss ⟹ 2 dof per node ⟹ 4 dof per element

Keep rows#/cols# 1 & 7 of 3-D Frame member stiffness matrix $\Rightarrow \left[k_L^{(e)}\right] = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

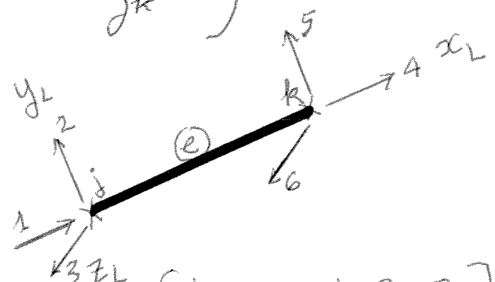
or $\left[k_L^{(e)}\right] = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\left[C^{(e)}\right] = \begin{bmatrix} c_x & c_y \\ -c_y & c_x \end{bmatrix}$ where $c_x = \frac{x_k - x_j}{L_{jk}^{(e)}}$, $c_y = \frac{y_k - y_j}{L_{jk}^{(e)}}$ ------ (2.43)

$$\underset{4\times4}{\left[k_G^{(e)}\right]} = \underset{4\times4}{[R]^T} \underset{4\times4}{\left[k_L^{(e)}\right]} \underset{4\times4}{[R]}$$

$$\underset{4\times4}{[R]} = \begin{bmatrix} \boxed{C} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \boxed{C} \end{bmatrix}$$

or $\left[k_G^{(e)}\right] = \left(\frac{AE}{L}\right)\begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix}$ ------ (2.20)



3-D Truss ⟹ 3 dof per node ⟹ 6 dof per element

Keep rows/cols # 1 & 7 of 3-D Frame member stiffness matrix $\Rightarrow \left[k_L^{(e)}\right] = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow$ or $\left[k_L^{(e)}\right] = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

For a "General 3-D truss" orientation, then $\underset{3\times3}{[C]} = \begin{bmatrix} c_x & c_y & c_z \\ \frac{-c_x c_y}{c_{xz}} & c_{xz} & \frac{-c_y c_z}{c_{xz}} \\ \frac{-c_z}{c_{xz}} & 0 & \frac{c_x}{c_{xz}} \end{bmatrix}$ ------ (2.41) or Eq(4.86) in G&W Book

For a "Vertical 3-D truss" orientation, then: parallel to Y_{Global}

$$[C] = \begin{bmatrix} 0 & c_y & 0 \\ -c_y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{------ (2.42)}$$

, where: $c_z = \frac{z_k - z_j}{L_{jk}^{(e)}}$, $c_{xz} = \sqrt{c_x^2 + c_z^2}$ ------ (2.43)

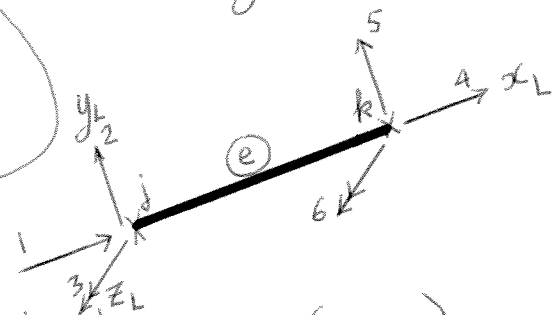
$$[R] = \underset{6\times6}{\begin{bmatrix} \boxed{\underset{3\times3}{C}} & \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} & \boxed{\underset{3\times3}{C}} \end{bmatrix}}$$

and $\underset{6\times6}{\left[k_G^{(e)}\right]} = [R]^T \left[k_L^{(e)}\right] [R]$

$$= \left(\frac{EA_x}{L}\right)\begin{bmatrix} \boxed{\begin{matrix} c_x^2 & c_x c_y & c_x c_z \\ c_y c_x & c_y^2 & c_y c_z \\ c_z c_x & c_z c_y & c_z^2 \end{matrix}} & \boxed{-c} \\ \boxed{-c} & \end{bmatrix}$$

2-D {Beam / Frame (include axial deformation) ⇒ 3 dof per node, 6 dof per element

Keep rows#/cols# $\boxed{1, 2, 6}$; $\boxed{7, 8, 12}$ of 3-D Frame member stiffness matrix



$$\underset{6\times 6}{\left[k_L^{(e)}\right]} = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & & & & & \\ \times & & 6\times 6 & & & \\ \times & & & & & \\ \times & & & & & \\ \times & & & & & \end{bmatrix} = \text{see Eq. just above Eq. (2.44)}$$

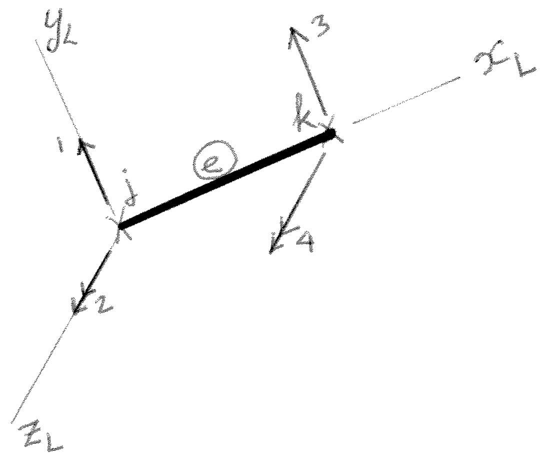
$$\underset{3\times 3}{[C]} = \begin{bmatrix} c_x & c_y & 0 \\ -c_y & c_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{---- (2.44)}$$

$$\underset{6\times 6}{[R]} = \begin{bmatrix} \boxed{\underset{3\times 3}{C}} & \boxed{\underset{3\times 3}{O}} \\ \boxed{\underset{3\times 3}{O}} & \boxed{\underset{3\times 3}{C}} \end{bmatrix} \quad \text{----- (2.45)}$$

$$\left[k_G^{(e)}\right] = \underbrace{[R]^T \left[k_L^{(e)}\right] [R]}_{\text{see Table 2.2 of ODU web-site}} \quad \text{---- (2.46)}$$

2-D {Frame / Beam (NEGLECT axial deformation) ⇒ 2 dof per node, 4 dof per element

Keep rows#/cols# $\boxed{2, 6}$; $\boxed{8, 12}$ of 3-D Frame member stiffness matrix



Hence:

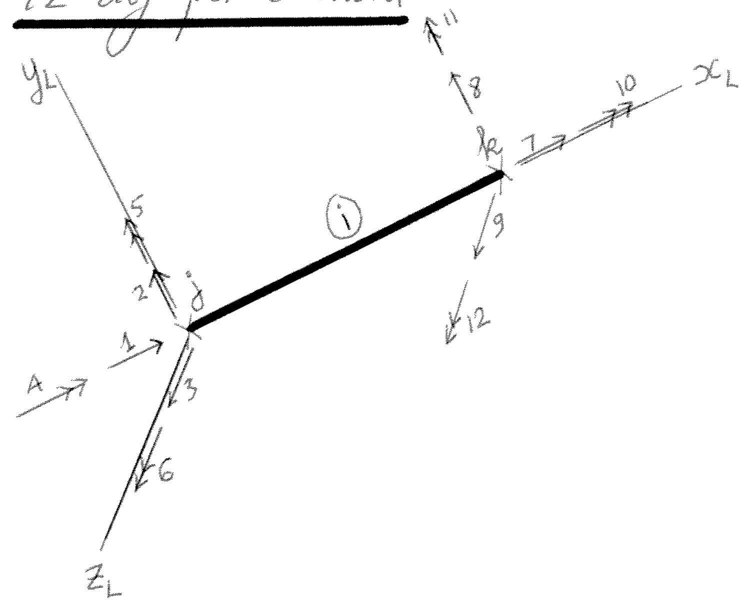
$$\underset{4\times 4}{\left[k_L^{(e)}\right]} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \times & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ \times \ \text{SYM} & \times & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \times & \times & \times & \frac{4EI_z}{L} \end{bmatrix} = \left[k_G^{(e)}\right] \text{ for 2-D } \underline{\underline{\text{Beam}}}$$

$$= \underset{4\times 4}{\left[k_G^{(e)}\right]}$$

2-D Frame Element Stiffness Matrix, shown in Eq. (2.46) without rows/cols # 1 & 4

For 3-D (Space) Frame $\Rightarrow$ 6 dof per node

12 dof per element



1	T_x	F_x	At Node j
2	T_y	F_y	
3	T_z	F_z	
4	θ_x	$M_x \equiv T_x$	
5	θ_y	M_y	
6	θ_z	M_z	
7	T_x	F_x	At Node k
8	T_y	F_y	
9	T_z	F_z	
10	θ_x	$M_x \equiv T_x$	
11	θ_y	M_y	
12	θ_z	M_z	

$$\underset{12\times12}{[R]} = \begin{bmatrix} \boxed{C} & \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{C} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{C} & \boxed{0} \\ \boxed{0} & \boxed{0} & \underbrace{\boxed{0}}_{3\times3} & \underbrace{\boxed{C}}_{3\times3} \end{bmatrix}$$

For a "General 3-D Frame" orientation,

$$\underset{3\times3}{[C]} = \begin{bmatrix} C_x & C_y & C_z \\ \dfrac{\{-C_x C_y \cos(\alpha) - C_z \sin(\alpha)\}}{C_{xz}} & C_{xz}\cos(\alpha) & \dfrac{\{-C_y C_z \cos(\alpha) + C_x \sin(\alpha)\}}{C_{xz}} \\ \dfrac{\{C_x C_y \sin(\alpha) - C_z \cos(\alpha)\}}{C_{xz}} & -C_{xz}\sin(\alpha) & \dfrac{\{C_y C_z \sin(\alpha) + C_x \cos(\alpha)\}}{C_{xz}} \end{bmatrix}$$

Eq.(4.93)

Note: If $\alpha = 0$, then Eq.(4.93) become same as Eq.(2.41) of a 3-D Truss!

where:

$$\left.\begin{aligned} x_{ps} &= x_p - x_j \\ y_{ps} &= y_p - y_j \\ z_{ps} &= z_p - z_j \end{aligned}\right\} \quad \text{.....(4.96)}$$

Note: (x_p, y_p, z_p) = coordinates of any point "p" on the x_L & y_L plane!!

$$\left.\begin{aligned} x_{p\gamma} &= C_x x_{ps} + C_y y_{ps} + C_z z_{ps} \\ y_{p\gamma} &= \frac{-C_x C_y}{C_{xz}} x_{ps} + C_{xz} y_{ps} - \frac{C_y C_z}{C_{xz}} z_{ps} \\ z_{p\gamma} &= \frac{-C_z}{C_{xz}} x_{ps} + \frac{C_x}{C_{xz}} z_{ps} \end{aligned}\right\} \quad \text{.....(4.97)}$$

$$\sin(\alpha) = \frac{z_{p\gamma}}{\sqrt{y_{p\gamma}^2 + z_{p\gamma}^2}} \quad \text{and} \quad \cos(\alpha) = \frac{y_{p\gamma}}{\sqrt{y_{p\gamma}^2 + z_{p\gamma}^2}} \quad \text{.....(4.98)}$$

For a VERTICAL 3-D (space) frame orientation, then

$$\underset{3\times3}{[C_{vert.}]} = \begin{bmatrix} 0 & C_y & 0 \\ -C_y\cos(\alpha) & 0 & \sin(\alpha) \\ C_y\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \qquad (4.99)$$

where:

$$\left.\begin{aligned} \sin(\alpha) &= \frac{z_{ps}}{\sqrt{x_{ps}^2 + z_{ps}^2}} \\ \cos(\alpha) &= \frac{-x_{ps}}{\sqrt{x_{ps}^2 + z_{ps}^2}}\, C_y \end{aligned}\right\} \qquad (4.100)$$

References

Matrix Analysis of Framed Structures, Second Edition,
Weaver and Gere (Van Nostrand Reinhold Company, 1980)
publisher
ISBN # 0-442-25773-2