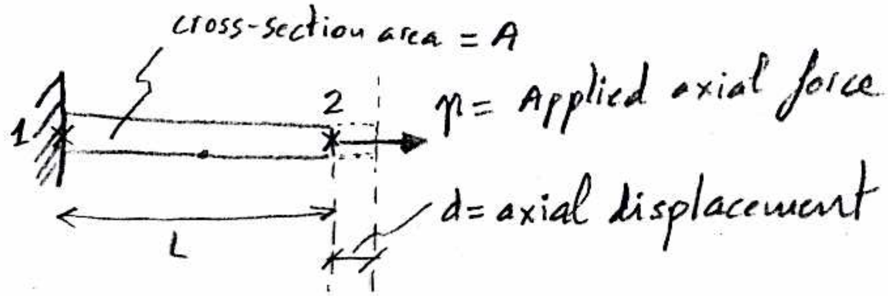
Domain Decomposition (1-D Axial Rod)

(A) Theory (Force-Deformation Relationship)



$$d = \frac{rL}{AE}$$

E → Young Modulus

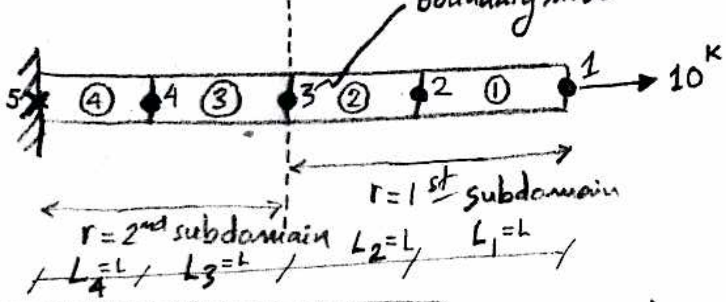
Hence, for 1 axial rod element : $\boxed{\left(\frac{AE}{L}\right) * d = r}$ → at "node" 2

for each "rod element" or $[k^{(e)}]_{\substack{1\times1 \\ N\times N}} * d^{(e)}_{\substack{1\times1 \\ N\times1}} = r^{(e)}_{\substack{1\times1 \\ N\times1}}$

Finite Element stiffness Matrix

$$\left(\frac{AE}{L}\right) * \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

(B) Structural Example (sub-domain formulation)



Assume: $A = E = L = 1$

For r = 1st subdomain

$$[K^{(r=1)}]_{3\times3} = \sum_{e=1}^{2} [k^{(e)}]_{2\times2}$$

where

$$k^{(e=1)} = \left(\frac{AE}{L}\right) \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \hline & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$k^{(e=2)} = \left(\frac{AE}{L}\right) \begin{array}{c|c} & \begin{array}{cc} 2 & 3 \end{array} \\ \hline & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

So:

$$[K^{(r=1)}]_{3\times3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & \\ -1 & 1+1 & -1 \\ & -1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & -1 \\ \hline 0 & -1 & 1 \end{array}\right] \end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \left[\begin{array}{c|c} K_{II}^{(r=1)} & K_{IB}^{(r=1)} \\ \hline K_{BI}^{(r=1)} & K_{BB}^{(r=1)} \end{array}\right]$$

$$\left\{\vec{P}^{(r=1)}\right\}_{3\times1} = \begin{array}{c} \\ \end{array}\left\{\begin{array}{c} 10^K \\ 0^K \\ 0^K \end{array}\right\}\begin{array}{c} 1 \\ 2 \\ 3 \end{array} = \left\{\begin{array}{c} 10^K \\ 0^K \\ \hline 0^K \end{array}\right\} = \left\{\begin{array}{c} \vec{P}_I^{(r=1)} \\ \hline \vec{P}_B^{(r=1)} \end{array}\right\}$$

Thus:

$$\left[K^{(r=1)}\right]_{3\times3}\left\{\vec{D}^{(r=1)}\right\}_{3\times1} = \left\{\vec{P}^{(r=1)}\right\}_{3\times1}$$

or

$$\begin{bmatrix} K_{II}^{(r=1)} & K_{IB}^{(r=1)} \\ K_{BI}^{(r=1)} & K_{BB}^{(r=1)} \end{bmatrix} * \begin{Bmatrix} D_I^{(r=1)} \\ D_B^{(r=1)} \end{Bmatrix} = \begin{Bmatrix} P_I^{(r=1)} \\ P_B^{(r=1)} \end{Bmatrix} \rightarrow K_{II}^{(r)}\vec{D}_I^{(r)} + K_{IB}^{(r)} D_B^{(r)} = P_I^{(r)}$$

so:

$$\underbrace{D_I^{(r)} = \left[K_{II}^{(r)}\right]^{-1} * \left\{P_I^{(r)} - K_{IB}^{(r)} D_B^{(r)}\right\}}_{\text{Eq. (20)}}$$

$$K_{BI}^{(r)} D_I^{(r)} + K_{BB}^{(r)} D_B^{(r)} = P_B^{(r)}$$

$$\left[K_{BB}^{(r)} - \underset{1\times2}{K_{BI}^{(r)}}\underset{2\times2}{\left[K_{II}^{(r)}\right]^{-1}}\underset{2\times1}{K_{IB}^{(r)}}\right]\vec{D}_B^{(r)} = \left\{P_B^{(r)} - K_{BI}^{(r)}\left[K_{II}^{(r)}\right]^{-1}\vec{P}_I^{(r=1)}\right\}$$

or

$$\underset{\substack{1\times1 \\ N\times N}}{\left[\bar{K}_B^{(r=1)}\right]} * \underset{\substack{1\times1 \\ N\times1}}{\vec{D}_B^{(r=1)}} = \underset{\substack{1\times1 \\ N\times1}}{\left\{\bar{P}_B^{(r=1)}\right\}}$$

Similarly, $\boxed{\text{for } r = 2^{nd} \text{ subdomain}}$

$$\left[k^{(e=3)}\right]_{2\times2} = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & 1 & -1 \\ 4 & -1 & 1 \end{array} * \cancel{\left(\frac{AE}{L}\right)} \nearrow 1$$

$$\left[k^{(e=4)}\right]_{2\times2} = \begin{array}{c|cc} & 4 & 5 \\ \hline 4 & 1 & -1 \\ 5 & -1 & 1 \end{array} * \cancel{\left(\frac{AE}{L}\right)} \nearrow 1$$

So:
$$\left[K^{(r=2)}\right]_{3\times 3} = \begin{array}{c|c} & \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & -1 & \\ -1 & 1+1 & -1 \\ & -1 & 1 \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 5 \end{matrix} & \left[\begin{array}{c|cc} 1 & -1 & 0 \\ \hline -1 & 2 & -1 \\ 0 & -1 & 1 \end{array}\right] \end{array}$$

$K_{BB}^{(r=2)}$ $K_{BI}^{(r=2)}$ $K_{IB}^{(r=2)}$ $K_{II}^{(r=2)}$

$$\left\{\vec{P}^{(r=2)}\right\}_{3\times 1} = \left\{\begin{array}{c} 0^K \\ \hline 0^K \\ 0^K \end{array}\right\} = \left\{\begin{array}{c} P_B^{(r=2)} \\ \hline P_I^{(r=2)} \end{array}\right\}$$

APLLYING Dirichlet Boundary Condition (Ex: $D_3 = 0$) $\Longrightarrow$
$$\left[K_{bc}^{(r=2)}\right] = \begin{array}{c|c} & \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 5 \end{matrix} & \left[\begin{array}{c|cc} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \qquad \left\{P_{bc}^{(r=2)}\right\} = \left\{\begin{array}{c} 0^K \\ 0^K \\ 0'' \end{array}\right\}$$

similarly:
$$\left[K_{BB}^{(r=2)} - \underset{1\times 2}{K_{BI}^{(r=2)}} \underset{2\times 2}{\left[K_{II}^{(r=2)}\right]^{-1}} \underset{2\times 1}{\left[K_{IB}^{(r=2)}\right]}\right] \vec{D}_B^{(r=2)} = \left\{P_B^{(r=2)} - K_{BI}^{(r)} \left[K_{II}^{(r)}\right]^{-1} P_I^{(r)}\right\}$$

or
$$\underset{\substack{N\times N \\ 1\times 1}}{\left[\bar{K}_B^{(r=2)}\right]} * \underset{\substack{N\times 1 \\ 1\times 1}}{\vec{D}_B^{(r=2)}} = \left\{\bar{P}_B^{(r=2)}\right\}_{\substack{N\times 1 \\ 1\times 1}}$$

Now, assembling 2 sub-domains

$$\bar{K}_B = \sum_{r=1}^{2} \bar{K}_B^{(r)} = \left[\bar{K}_B^{(r=1)}\right]_{1\times 1} + \left[\bar{K}_B^{(r=2)}\right]_{1\times 1}$$

$$\underset{\substack{N\times 1 \\ 1\times 1}}{\left\{\bar{P}_B\right\}} = \sum_{r=1}^{2} \bar{P}_B^{(r)}$$

Solve for boundary unknown(s) first $\left[\bar{K}_B\right]\left\{\overset{??}{\vec{D}_B}\right\}_{@\,\text{node } 3} = \left\{\bar{P}_B\right\}$ --- Eq. (10)

Then, solve for interior unknowns for 1st sub-domain (see Eq. 20)
also for 2nd subdomain (see Eq. 20)

ⓒ Same Example, conventional method

$$[K] = \sum_{e=1}^{4} k^{(e)} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \end{bmatrix} \end{array} \;\&\; \vec{P} = \left\{\begin{array}{c} 10^K \\ 0^K \\ 0^K \\ 0^K \\ 0^K \end{array}\right\} \Longrightarrow [K_{bc}] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & \cancel{-1} \\ 0 & 0 & 0 & \cancel{1+1} & \end{bmatrix} \;\&\; \vec{P}_{bc} = \left\{\begin{array}{c} 10^K \\ 0^K \\ 0^K \\ 0^K \\ 0 \end{array}\right\}$$

Therefore [illegible]