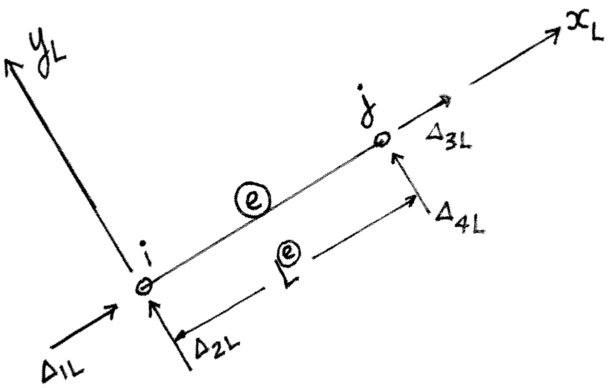
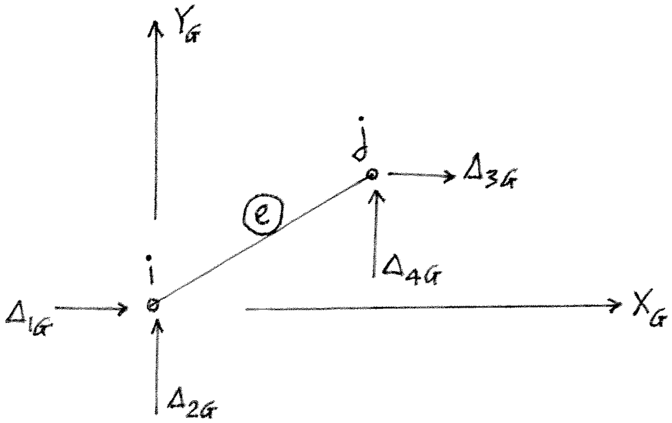
2-D Truss (2 dof per node, 4 dof per element)



$$k_L^{(e)} = \left(\frac{AE}{L^{(e)}}\right)\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\underset{4\times4}{k_G^{(e)}} = \underset{4\times4}{[R]^T}\,\underset{4\times4}{\left[k_L^{(e)}\right]}\,\underset{4\times4}{[R]}$$

where: $[R] = \begin{bmatrix} [C] & [0] \\ [0] & [C] \end{bmatrix}$

$$[C] = \begin{bmatrix} c_x & c_y \\ -c_y & c_x \end{bmatrix}$$

$c_y = \dfrac{y_j - y_i}{L^{(e)}}$

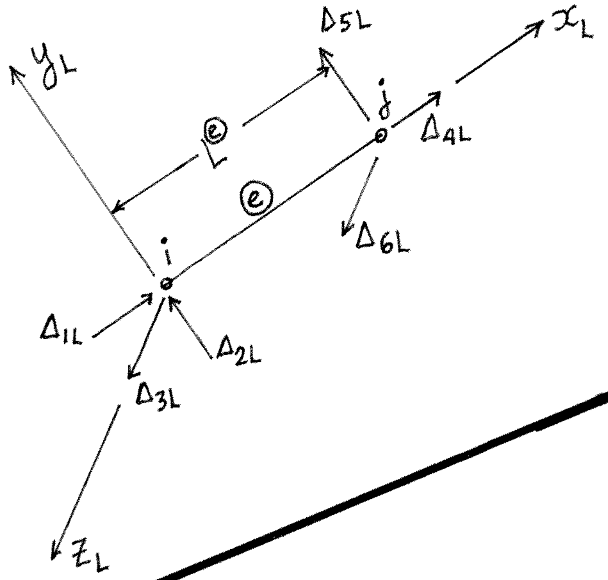
$c_x = \dfrac{x_j - x_i}{L^{(e)}}$

Hence:

$$\left[k_G^{(e)}\right] = \frac{EA}{L}\left[\begin{array}{cc|cc} C_X^2 & C_XC_Y & -C_X^2 & -C_XC_Y \\ C_XC_Y & C_Y^2 & -C_XC_Y & -C_Y^2 \\ \hline -C_X^2 & -C_XC_Y & C_X^2 & C_XC_Y \\ -C_XC_Y & -C_Y^2 & C_XC_Y & C_Y^2 \end{array}\right]$$

where $L = L^{(e)} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$

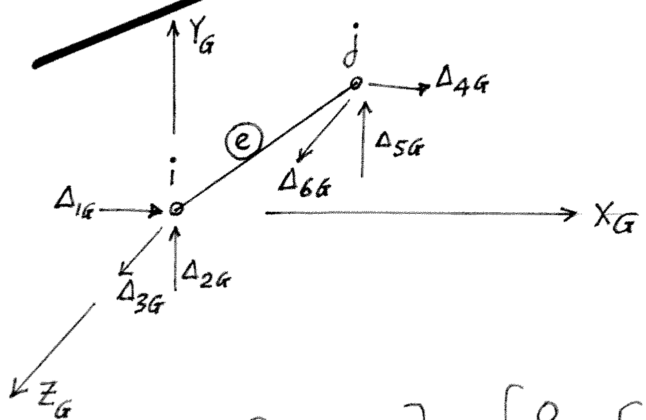
3-D Truss (3 dof per node, 6 dof per element)



$$k_L^{(e)} = \left(\frac{AE}{L^{(e)}}\right)\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_G^{(e)} = [R]^T \underset{6\times 6}{\left[k_L^{(e)}\right]} \underset{6\times 6}{[R]}$$

where: $[R] = \begin{bmatrix} [C] & [0] \\ [0] & [C] \end{bmatrix}$



$\dfrac{Z_j - Z_i}{L^{(e)}}$

$$\underset{3\times 3}{[C]} = \begin{bmatrix} C_x & C_y & C_z \\ \dfrac{-C_x C_y}{C_{xz}} & C_{xz} & \dfrac{-C_y C_z}{C_{xz}} \\ \dfrac{-C_z}{C_{xz}} & 0 & \dfrac{C_x}{C_{xz}} \end{bmatrix} = \text{General orientation member}$$

$\sqrt{C_x^2 + C_z^2}$

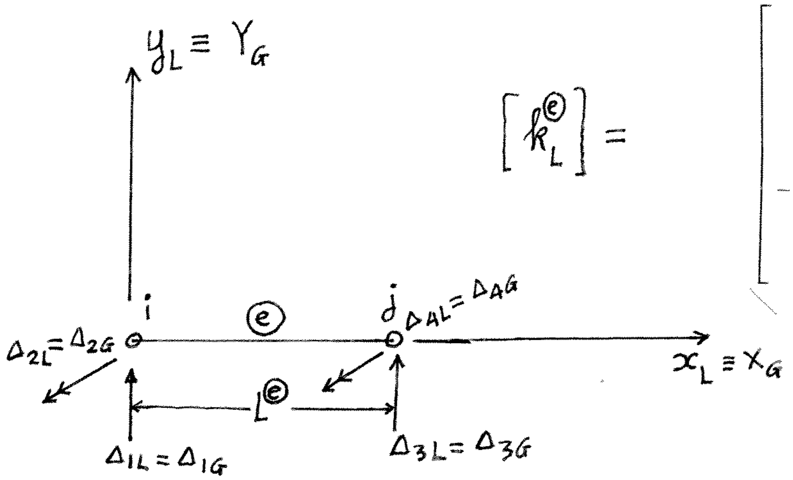
$$[C_{vertical}] = \begin{bmatrix} 0 & C_y & 0 \\ -C_y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{vertical member (parallel to } Y_G \text{ axis)}$$

Hence:

$$\left[k_G^{(e)}\right] = \left(\frac{EA}{L}\right)\left[\begin{array}{ccc|ccc} C_X^2 & C_Y C_X & C_Z C_X & -C_X^2 & -C_Y C_X & -C_Z C_X \\ C_X C_Y & C_Y^2 & C_Z C_Y & -C_X C_Y & -C_Y^2 & -C_Z C_Y \\ C_X C_Z & C_Y C_Z & C_Z^2 & -C_X C_Z & -C_Y C_Z & -C_Z^2 \\ \hline -C_X^2 & -C_Y C_X & -C_Z C_X & C_X^2 & C_Y C_X & C_Z C_X \\ -C_X C_Y & -C_Y^2 & -C_Z C_Y & C_X C_Y & C_Y^2 & C_Z C_Y \\ -C_X C_Z & -C_Y C_Z & -C_Z^2 & C_X C_Z & C_Y C_Z & C_Z^2 \end{array}\right]$$

where: $L = L^{(e)} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$

2-D Beam/~~Frame~~ (NEGLECT axial deformation: 2 dof per node, 4 dof per element)



$$
\left[k_L^{e}\right] = \left[\begin{array}{cc|cc}
\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
\hline
-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right]
$$

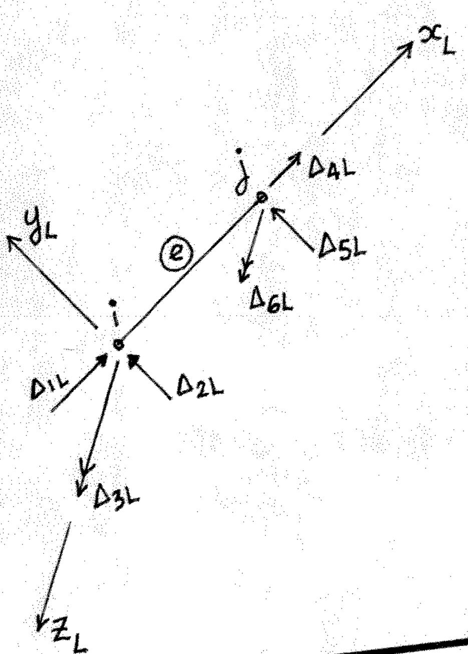
where: $L = L^{e} = (x_j - x_i)$

$$
\left[k_G^{e}\right] = \underset{4\times4}{[R]^T}\ \underset{4\times4}{\left[k_L^{e}\right]}\ \cancel{[R]} = \left[k_L^{e}\right]
$$

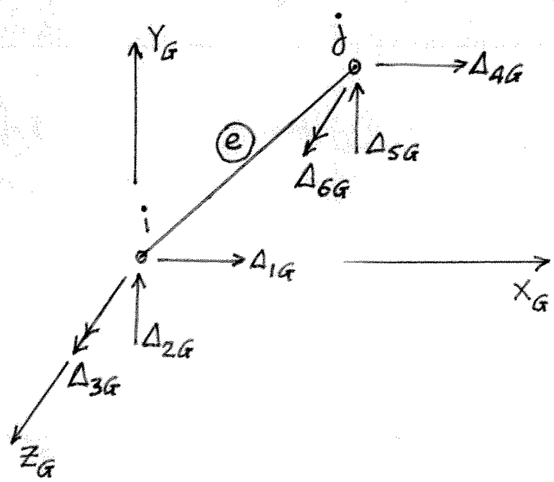
$$
\underset{4\times4}{[I]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

since $x_L = X_G$
$y_L = Y_G$

2-D Beam/Frame (INCLUDE axial deformation: 3 dof per node, 6 dof per element)



$$\underset{6\times 6}{\left[k_L^{(e)}\right]} = \left[\begin{array}{ccc|ccc} \frac{EA_X}{L} & 0 & 0 & -\frac{EA_X}{L} & 0 & 0 \\ 0 & \frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} & 0 & -\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} \\ 0 & \frac{6EI_Z}{L^2} & \frac{4EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} \\ \hline -\frac{EA_X}{L} & 0 & 0 & \frac{EA_X}{L} & 0 & 0 \\ 0 & -\frac{12EI_Z}{L^3} & -\frac{6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & -\frac{6EI_Z}{L^2} \\ 0 & \frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & \frac{4EI_Z}{L} \end{array}\right]$$



$$\underset{6\times 6}{\left[k_G^{(e)}\right]} = \underset{6\times 6}{[R]^T} \underset{6\times 6}{\left[k_L^{(e)}\right]} \underset{6\times 6}{[R]}$$

where: $$[R] = \begin{bmatrix} [C] & [0] \\ [0] & [C] \end{bmatrix}$$

$$\underset{3\times 3}{[C]} = \begin{bmatrix} c_x & c_y & 0 \\ -c_y & c_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c_y = (y_j - y_i)/L^{(e)}$

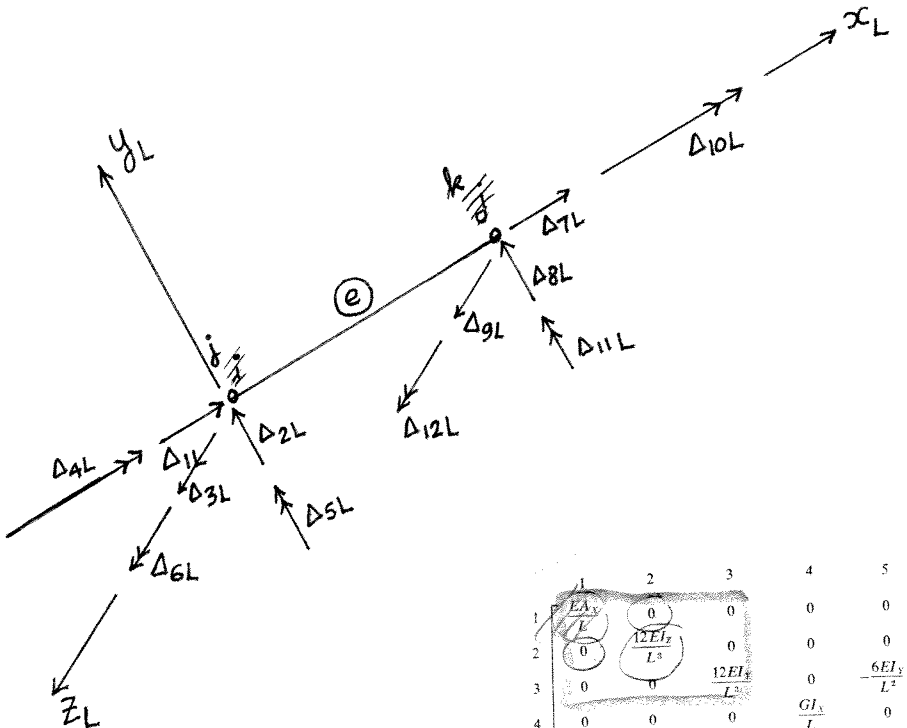
$c_x = (x_j - x_i)/L^{(e)}$

and $L^{(e)} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$

Hence,

$$\underset{6\times 6}{\left[k_G^{(e)}\right]} = \left[\begin{array}{ccc|ccc} \frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2 & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\frac{6EI_Z}{L^2}C_Y & -\left(\frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2\right) & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\frac{6EI_Z}{L^2}C_Y \\ \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2 & \frac{6EI_Z}{L^2}C_X & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\left(\frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2\right) & \frac{6EI_Z}{L^2}C_X \\ -\frac{6EI_Z}{L^2}C_Y & \frac{6EI_Z}{L^2}C_X & \frac{4EI_Z}{L} & \frac{6EI_Z}{L^2}C_Y & -\frac{6EI_Z}{L^2}C_X & \frac{2EI_Z}{L} \\ \hline -\left(\frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2\right) & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{6EI_Z}{L^2}C_Y & \frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2 & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{6EI_Z}{L^2}C_Y \\ -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\left(\frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2\right) & -\frac{6EI_Z}{L^2}C_X & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2 & -\frac{6EI_Z}{L^2}C_X \\ -\frac{6EI_Z}{L^2}C_Y & \frac{6EI_Z}{L^2}C_X & \frac{2EI_Z}{L} & \frac{6EI_Z}{L^2}C_Y & -\frac{6EI_Z}{L^2}C_X & \frac{4EI_Z}{L} \end{array}\right]$$

3-D Space Frame (6 dof per node, 12 dof per element)



$$\left[k_L^{(e)}\right]_{12\times 12} = \begin{array}{c|cccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
1 & \frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} & 0 & -\frac{12EI_Z}{L^3} & 0 & 0 & -0 & \frac{6EI_Z}{L^2} \\
3 & 0 & 0 & \frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0 & 0 & 0 & -\frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0 \\
4 & 0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0 \\
5 & 0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 \\
6 & 0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} \\
7 & -\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & -\frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} \\
9 & 0 & 0 & -\frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 & 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 \\
10 & 0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0 \\
11 & 0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 \\
12 & 0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L}
\end{array}$$

$$\underset{12\times12}{\left[k_G^{(e)}\right]} = \underset{12\times12}{[R]^T} \underset{12\times12}{\left[k_L^{(e)}\right]} \underset{12\times12}{[R]} \quad ; \text{ where: } [R] = \begin{bmatrix} [c] & [0] & [0] & [0] \\ [0] & [c] & [0] & [0] \\ [0] & [0] & [c] & [0] \\ [0] & [0] & [0] & [c] \end{bmatrix} \quad \ldots\ldots (2.47)$$

$$[c] = \begin{bmatrix} C_x & C_y & C_z \\ \left\{-C_x C_y \cos(\alpha) - C_z \sin(\alpha)\right\} / \sqrt{C_x^2 + C_z^2} & \left(\sqrt{C_x^2 + C_z^2}\right)\cos(\alpha) & \dfrac{-C_y C_z \cos(\alpha) + C_x \sin(\alpha)}{\sqrt{C_x^2 + C_z^2}} \\ \left\{C_x C_y \sin(\alpha) - C_z \cos(\alpha)\right\} / \sqrt{C_x^2 + C_z^2} & \left(-\sqrt{C_x^2 + C_z^2}\right)\sin(\alpha) & \dfrac{C_y C_z \sin(\alpha) + C_x \cos(\alpha)}{\sqrt{C_x^2 + C_z^2}} \end{bmatrix} \quad \ldots\ldots (2.48)$$

$$x_{ps} = x_p - x_j \quad ; \quad y_{ps} = y_p - y_j \quad ; \quad z_{ps} = z_p - z_j \quad \ldots\ldots\ldots (2.50)$$

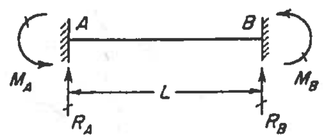
$$\sin(\alpha) = \frac{z_{ps}}{\sqrt{x_{ps}^2 + z_{ps}^2}} \quad ; \quad \cos(\alpha) = \frac{-x_{ps}}{\sqrt{x_{ps}^2 + z_{ps}^2}} * C_y \quad \ldots\ldots\ldots (2.51)$$

and point "p" can be any point located on the principal plane cross-section [see Fig. 2.7]

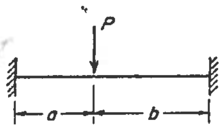
Introduction to Stiffness Matrix Method

Table A.1

Fixed-End Actions Caused by Loads



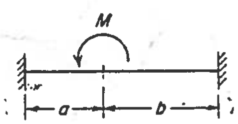
1



$M_A = \dfrac{Pab^2}{L^2}$ $\quad M_B = -\dfrac{Pa^2b}{L^2}$

$R_A = \dfrac{Pb^2}{L^3}(3a+b)$ $\quad R_B = \dfrac{Pa^2}{L^3}(a+3b)$

2

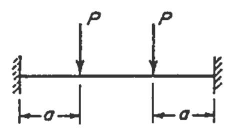


$M_A = \dfrac{Mb}{L^2}(2a-b)$

$M_B = \dfrac{Ma}{L^2}(2b-a)$

$R_A = -R_B = \dfrac{6Mab}{L^3}$

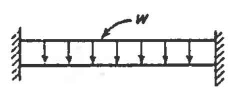
5



$M_A = -M_B = \dfrac{Pa}{L}(L-a)$

$R_A = R_B = P$

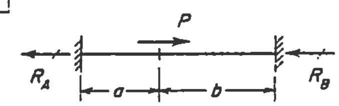
6



$M_A = -M_B = \dfrac{wL^2}{12}$

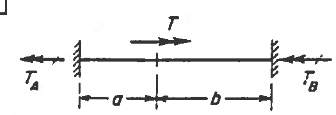
$R_A = R_B = \dfrac{wL}{2}$

3



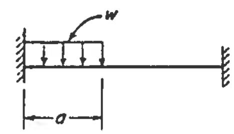
$R_A = \dfrac{Pb}{L}$ $\quad R_B = \dfrac{Pa}{L}$

4



$T_A = \dfrac{Tb}{L}$ $\quad T_B = \dfrac{Ta}{L}$

7



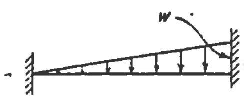
$M_A = \dfrac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$

$M_B = -\dfrac{wa^3}{12L^2}(4L - 3a)$

$R_A = \dfrac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$

$R_B = \dfrac{wa^3}{2L^3}(2L - a)$

8



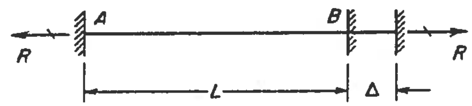
$M_A = \dfrac{wL^2}{30}$ $\quad M_B = -\dfrac{wL^2}{20}$

$R_A = \dfrac{3wL}{20}$ $\quad R_B = \dfrac{7wL}{20}$

Table A.2

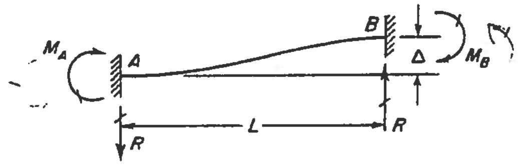
Fixed-End Actions Caused by End-Displacements

1



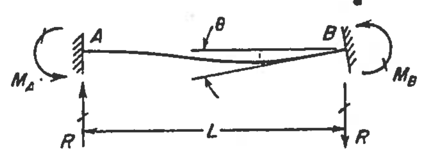
$$R = \frac{EA\Delta}{L}$$

2A



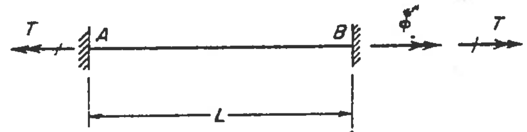
$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$

3A



$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$

4

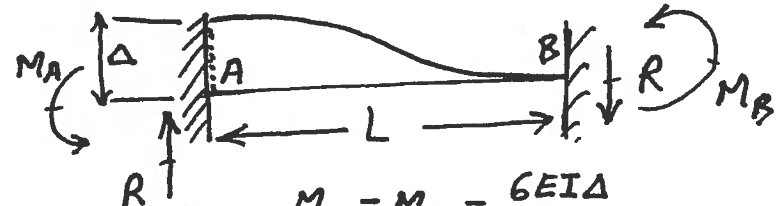


$$T = \frac{GJ\phi}{L}$$

G = shear modulus of elasticity

J = torsion constant

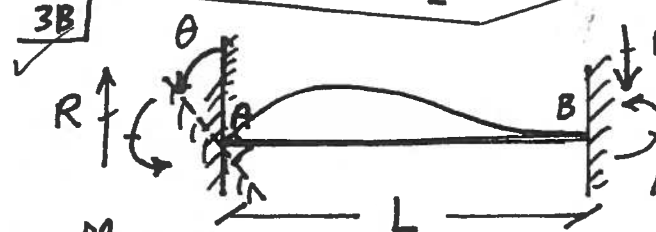
2B



$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R = \frac{12EI\Delta}{L^3}$$

3B



$$R = \frac{6EI\theta}{L^2}$$

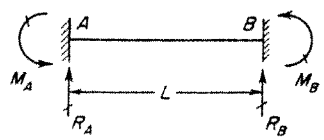
$$M_B = \frac{2EI\theta}{L}$$

$$M_A = \frac{4EI\theta}{L}$$

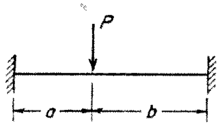
Introduction to Stiffness Matrix Method

Table A.1

Fixed-End Actions Caused by Loads



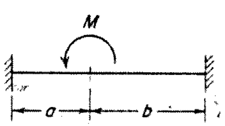
1



$$M_A = \frac{Pab^2}{L^2} \quad M_B = -\frac{Pa^2b}{L^2}$$

$$R_A = \frac{Pb^2}{L^3}(3a+b) \quad R_B = \frac{Pa^2}{L^3}(a+3b)$$

2

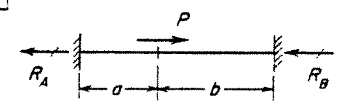


$$M_A = \frac{Mb}{L^2}(2a-b)$$

$$M_B = \frac{Ma}{L^2}(2b-a)$$

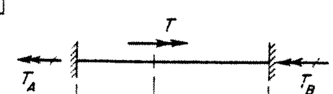
$$R_A = -R_B = \frac{6Mab}{L^3}$$

3



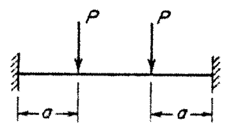
$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

4



$$T_A = \frac{Tb}{L} \qquad T_B = \frac{Ta}{L}$$

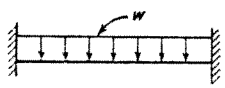
5



$$M_A = -M_B = \frac{Pa}{L}(L-a)$$

$$R_A = R_B = P$$

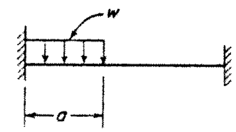
6



$$M_A = -M_B = \frac{wL^2}{12}$$

$$R_A = R_B = \frac{wL}{2}$$

7



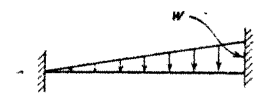
$$M_A = \frac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = -\frac{wa^3}{12L^2}(4L - 3a)$$

$$R_A = \frac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \frac{wa^3}{2L^3}(2L - a)$$

8



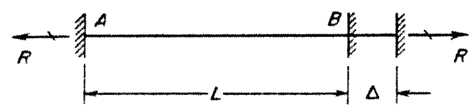
$$M_A = \frac{wL^2}{30} \qquad M_B = -\frac{wL^2}{20}$$

$$R_A = \frac{3wL}{20} \qquad R_B = \frac{7wL}{20}$$

Table A.2

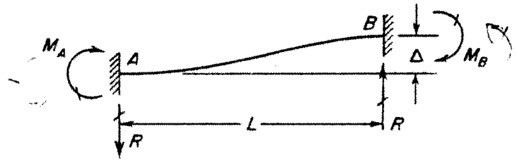
Fixed-End Actions Caused by End-Displacements

1



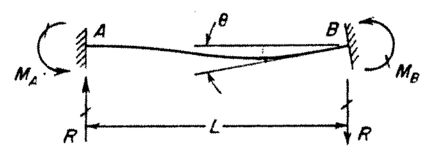
$$R = \frac{EA\Delta}{L}$$

2A



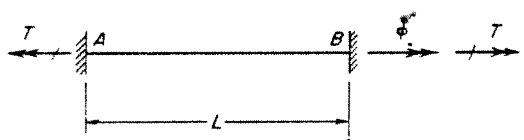
$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$

3A



$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$

4

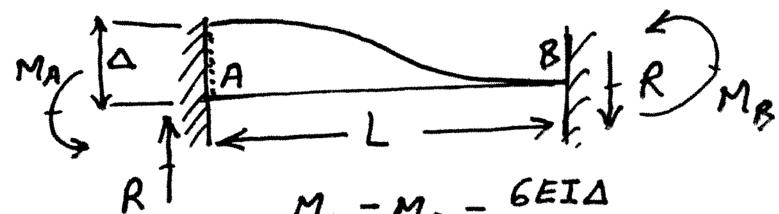


$$T = \frac{GJ\phi}{L}$$

G = shear modulus of elasticity

J = torsion constant

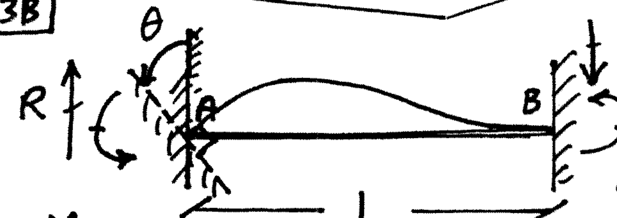
2B



$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R = \frac{12EI\Delta}{L^3}$$

3B

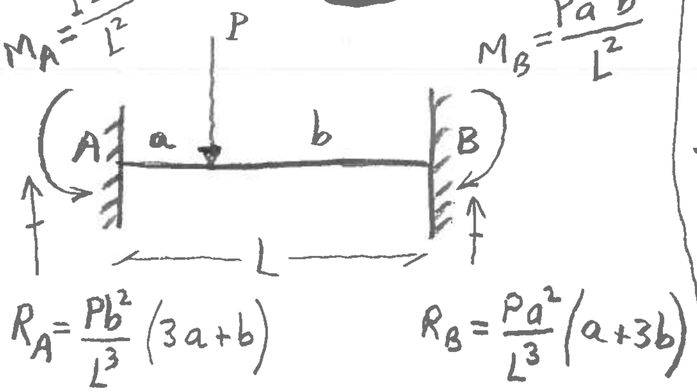


$$R = \frac{6EI\theta}{L^2}$$

$$M_B = \frac{2EI\theta}{L}$$

$$M_A = \frac{4EI\theta}{L}$$

Figure 1

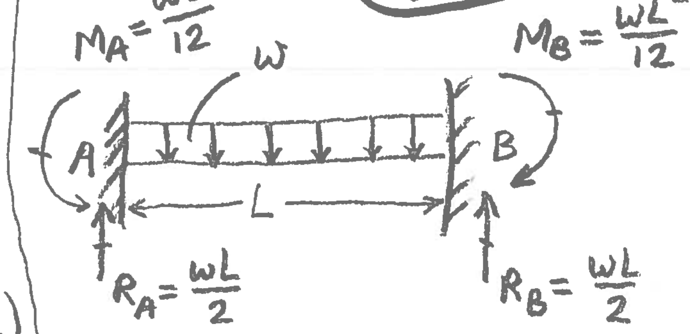


If $a = b = \frac{L}{2}$

Then $|M_A| = |M_B| = \frac{PL}{8}$

and $|R_A| = |R_B| = \frac{P}{2}$

Figure 2



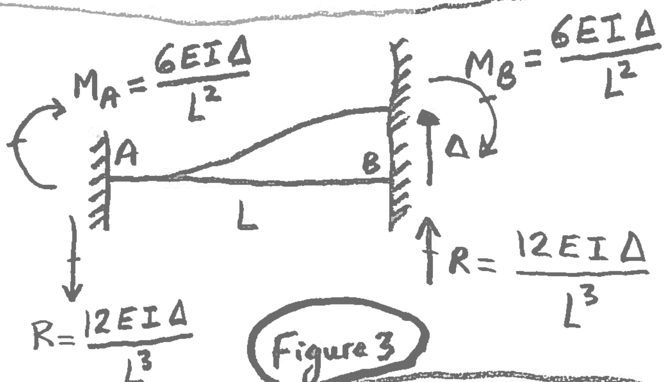
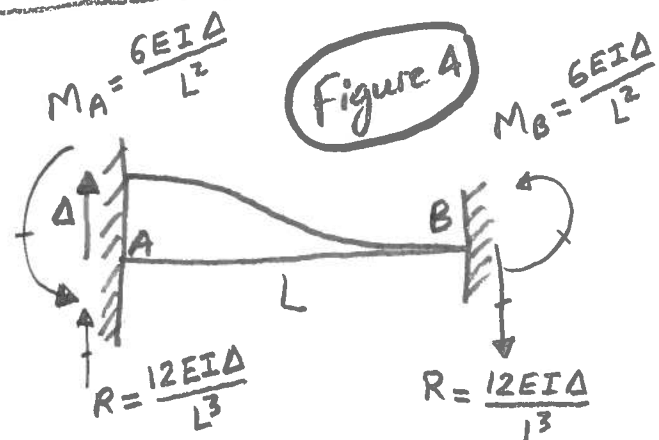


Figure 3

Figure 4



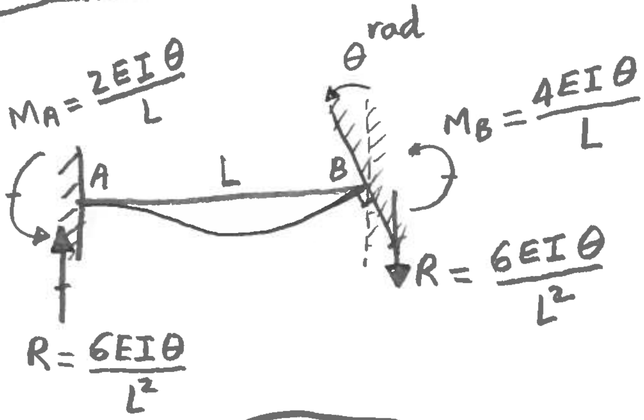


Figure 5

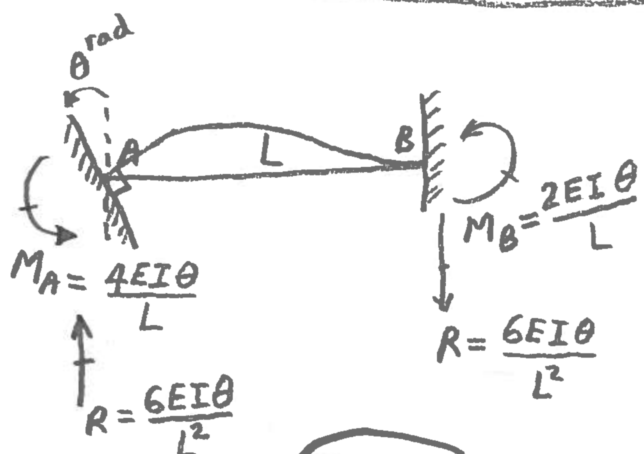


Figure 6

Fixed End <u>Actions</u> Caused by Applied Loads & by Settlements

Actions → Moment, Force

STIFFNESS METHOD

The following formulas are "always true/correct" for any element types →
- 2-D/3-D Truss Element
- 2-D Beam/Frame Element
- 3-D Beam/Frame Element

$$\left[k_G^{(e)}\right] = [R]^T \left[k_L^{(e)}\right] [R] \qquad (1)$$

where:

$$[R] = \begin{bmatrix} [C] & [0] \\ [0] & [C] \end{bmatrix} \qquad (2)$$

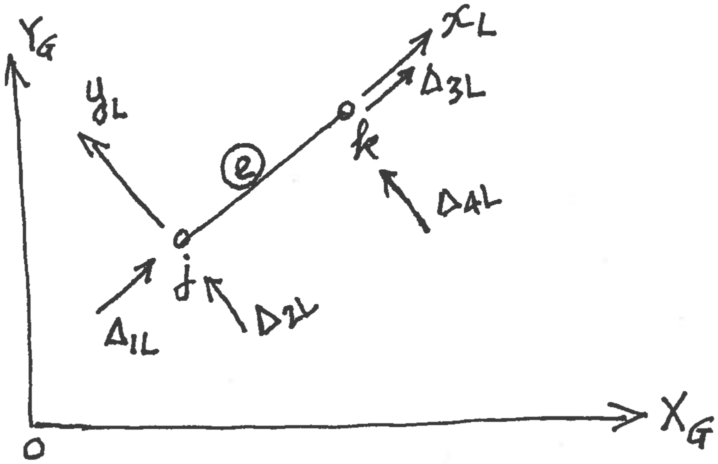
$$[C] = c_x,\ c_y,\ c_z,\ c_{xz},\ \text{etc....} \qquad (3)$$

$$c_{xz} = \sqrt{c_x^2 + c_z^2} \qquad (4)$$

$$\left.\begin{aligned} c_x &= \frac{x_k - x_j}{L_{jk}} \\ c_y &= \frac{y_k - y_j}{L_{jk}} \\ c_z &= \frac{z_k - z_j}{L_{jk}} \end{aligned}\right\} \qquad (5)$$

$$L_{jk} = \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2} \qquad (6)$$

2-D Truss

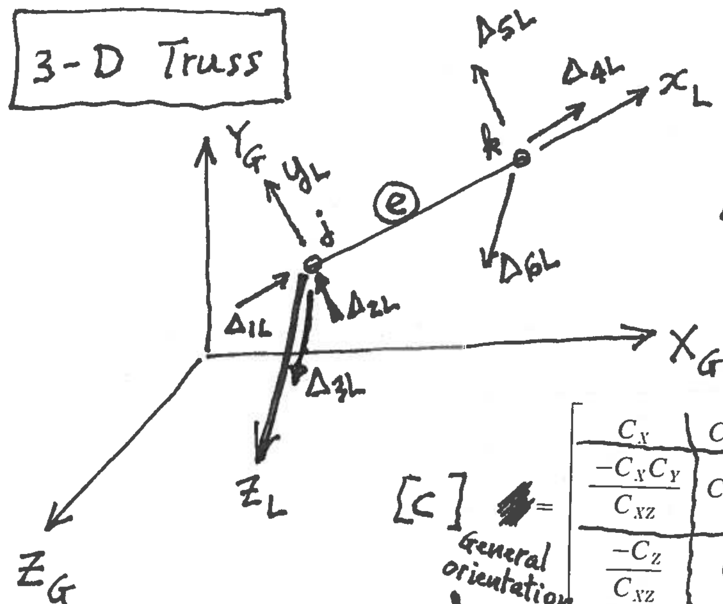


$$\frac{EA_X}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[k_L^e\right]$$

$$[C] = \begin{bmatrix} C_x & C_y \\ -C_y & C_x \end{bmatrix}$$

$$\frac{EA_X}{L}\left[\begin{array}{cc|cc} C_X^2 & C_XC_Y & -C_X^2 & -C_XC_Y \\ C_XC_Y & C_Y^2 & -C_XC_Y & -C_Y^2 \\ \hline -C_X^2 & -C_XC_Y & C_X^2 & C_XC_Y \\ -C_XC_Y & -C_Y^2 & C_XC_Y & C_Y^2 \end{array}\right] \checkmark = \left[k_G^e\right] = [R]^T\left[k_L^e\right][R]$$

3-D Truss



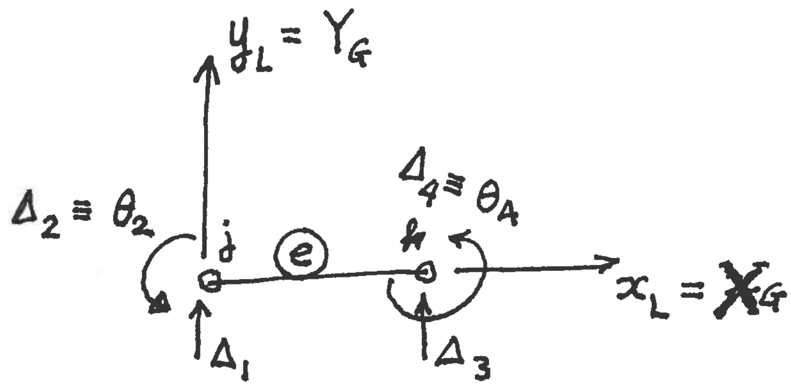
$$\left[k_L^e\right] = \frac{EA_X}{L}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$[C]_{\text{General orientation}} = \begin{bmatrix} C_X & C_Y & C_Z \\ \dfrac{-C_XC_Y}{C_{XZ}} & C_{XZ} & \dfrac{-C_YC_Z}{C_{XZ}} \\ \dfrac{-C_Z}{C_{XZ}} & 0 & \dfrac{C_X}{C_{XZ}} \end{bmatrix} \qquad [C]_{\text{vertical member}} = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

parallel to Y_G axis

$$\left[k_G^e\right] = \frac{EA_X}{L}\left[\begin{array}{ccc|ccc} C_X^2 & C_YC_X & C_ZC_X & -C_X^2 & -C_YC_X & -C_ZC_X \\ C_XC_Y & C_Y^2 & C_ZC_Y & -C_XC_Y & -C_Y^2 & -C_ZC_Y \\ C_XC_Z & C_YC_Z & C_Z^2 & -C_XC_Z & -C_YC_Z & -C_Z^2 \\ \hline -C_X^2 & -C_YC_X & -C_ZC_X & C_X^2 & C_YC_X & C_ZC_X \\ -C_XC_Y & -C_Y^2 & -C_ZC_Y & C_XC_Y & C_Y^2 & C_ZC_Y \\ -C_XC_Z & -C_YC_Z & -C_Z^2 & C_XC_Z & C_YC_Z & C_Z^2 \end{array}\right] = [R]^T\left[k_L^e\right][R]$$

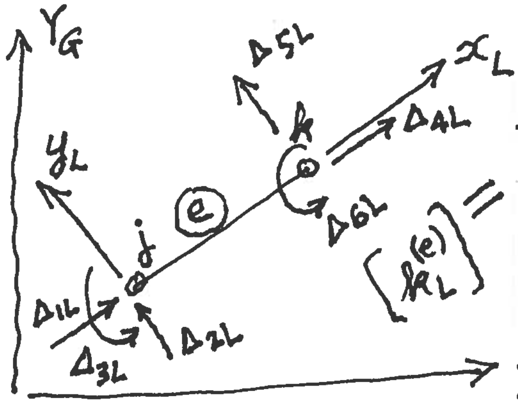
$\boxed{\text{2-D Beam}}$ (Neglect Axial)



$$[k_L^e] = \left[\begin{array}{cc|cc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ \hline -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] = [k_G^e] = [R]^T [k_L^e][R]$$

$$[I] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2-D Beam/Frame (Include Axial)



$$
\left[k_L^{(e)}\right] = \left[\begin{array}{ccc|ccc}
\frac{EA_X}{L} & 0 & 0 & -\frac{EA_X}{L} & 0 & 0 \\
0 & \frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} & 0 & -\frac{12EI_Z}{L^3} & \frac{6EI_Z}{L^2} \\
0 & \frac{6EI_Z}{L^2} & \frac{4EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} \\
\hline
-\frac{EA_X}{L} & 0 & 0 & \frac{EA_X}{L} & 0 & 0 \\
0 & -\frac{12EI_Z}{L^3} & -\frac{6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & -\frac{6EI_Z}{L^2} \\
0 & \frac{6EI_Z}{L^2} & \frac{2EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & \frac{4EI_Z}{L}
\end{array}\right]
$$

$$
\begin{bmatrix} C_X & C_Y & 0 \\ -C_Y & C_X & 0 \\ 0 & 0 & 1 \end{bmatrix} = [C]
$$

3x3 sub-matrices

$$
\begin{bmatrix} \boxed{C} & \boxed{0} \\ \boxed{0} & \boxed{C} \end{bmatrix}
$$

$$
\underset{6\times 6}{\left[k_G^e\right]} = \underset{6\times 6}{[R]^T}\ \underset{6\times 6}{\left[k_L^e\right]}\ \underset{6\times 6}{[R]}
$$

$$
\left[k_G^e\right] = \left[\begin{array}{ccc|ccc}
\frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2 & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\frac{6EI_Z}{L^2}C_Y & -\left(\frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2\right) & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\frac{6EI_Z}{L^2}C_Y \\
\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2 & \frac{6EI_Z}{L^2}C_X & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\left(\frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2\right) & \frac{6EI_Z}{L^2}C_X \\
-\frac{6EI_Z}{L^2}C_Y & \frac{6EI_Z}{L^2}C_X & \frac{4EI_Z}{L} & \frac{6EI_Z}{L^2}C_Y & -\frac{6EI_Z}{L^2}C_X & \frac{2EI_Z}{L} \\
\hline
-\left(\frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2\right) & -\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{6EI_Z}{L^2}C_Y & \frac{EA_X}{L}C_X^2 + \frac{12EI_Z}{L^3}C_Y^2 & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{6EI_Z}{L^2}C_Y \\
-\left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & -\left(\frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2\right) & -\frac{6EI_Z}{L^2}C_X & \left(\frac{EA_X}{L} - \frac{12EI_Z}{L^3}\right)C_XC_Y & \frac{EA_X}{L}C_Y^2 + \frac{12EI_Z}{L^3}C_X^2 & -\frac{6EI_Z}{L^2}C_X \\
-\frac{6EI_Z}{L^2}C_Y & \frac{6EI_Z}{L^2}C_X & \frac{2EI_Z}{L} & \frac{6EI_Z}{L^2}C_Y & -\frac{6EI_Z}{L^2}C_X & \frac{4EI_Z}{L}
\end{array}\right]
$$

"System" Equations

contributions from "Element" stiffness equations!! $[k_G^e]$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \Delta_1 ? \\ \Delta_2 ? \\ \Delta_3 ? \\ \Delta_4 = c_4 = 0'' \\ \Delta_5 = c_5 = 0'' \\ \Delta_6 = c_6 = 0'' \end{Bmatrix} = \begin{Bmatrix} 10^K \\ 0^K \\ 0^K \\ R_{2y} ? \\ R_{3x} ? \\ R_{3y} ? \end{Bmatrix} \quad \ldots\ldots (2.56)$$

"singular" matrix ☹

mixing knowns & unknowns ☹

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{31} & K_{32} & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} = \begin{Bmatrix} 10^K - K_{14}c_4 - K_{15}c_5 - K_{16}c_6 \\ 0^K - K_{24}c_4 - K_{25}c_5 - K_{26}c_6 \\ 0^K - K_{34}c_4 - K_{35}c_5 - K_{36}c_6 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \quad (2.60)$$

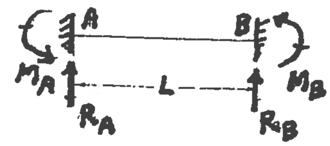
"Non-singular" ☺

RHS vector contains all known quantities ☺

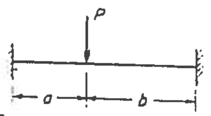
- Using Eq. (2.60) to solve $\Delta_1, \Delta_2, \ldots, \Delta_6$
- Then, using Eq. (2.56) to solve all unknown support reactions, such as R_{2y}, R_{3x} & R_{3y}

http://www.lions.odu.edu/~skadi002
Then SELECT/CLICK Educational & Assessment Tools
Then select/click Simultaneous Linear Equations (SLE)

Fixed-End Actions Caused by Loads



1



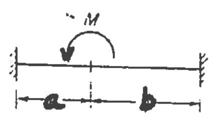
If $a = b = \frac{L}{2}$ $\quad \frac{PL}{8}$

$R_A = R_B = \frac{P}{2}$

$$M_A = \frac{Pab^2}{L^2} \qquad M_B = -\frac{Pa^2b}{L^2}$$

$$R_A = \frac{Pb^2}{L^3}(3a + b) \quad R_B = \frac{Pa^2}{L^3}(a + 3b)$$

2

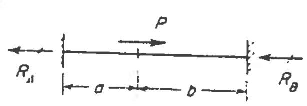


$$M_A = \frac{Mb}{L^2}(2a - b)$$

$$M_B = \frac{Ma}{L^2}(2b - a)$$

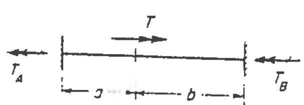
$$R_A = -R_B = \frac{6Mab}{L^3}$$

3



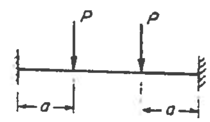
$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

4



$$T_A = \frac{Tb}{L} \; ; \qquad T_B = \frac{Ta}{L}$$

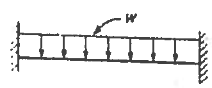
5



$$M_A = -M_B = \frac{Pa}{L}(L - a)$$

$$R_A = R_B = P$$

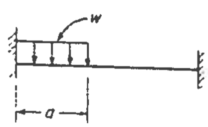
6



$$M_A = -M_B = \frac{wL^2}{12}$$

$$R_A = R_B = \frac{wL}{2}$$

7



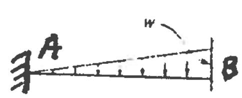
$$M_A = \frac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = -\frac{wa^3}{12L^2}(4L - 3a)$$

$$R_A = \frac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \frac{wa^3}{2L^3}(2L - a)$$

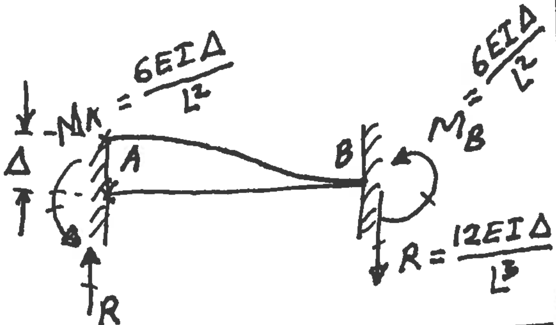
8

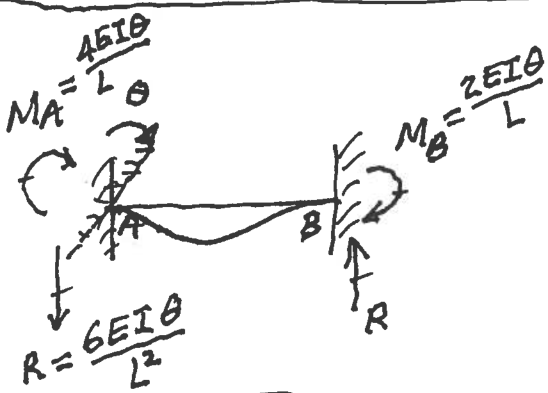


$$M_A = \frac{wL^2}{30} \; ; \quad M_B = -\frac{wL^2}{20}$$

$$R_A = \frac{3wL}{20} \; ; \quad R_B = \frac{7wL}{20}$$

page 1/2





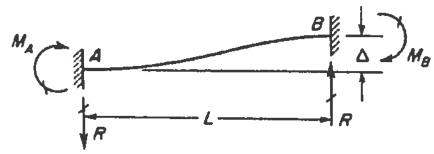
page 2/2

Fixed-End Actions Caused by End-Displacements

1

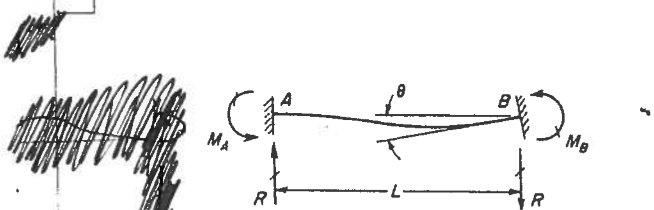
$$R = \frac{EA\Delta}{L}$$

2



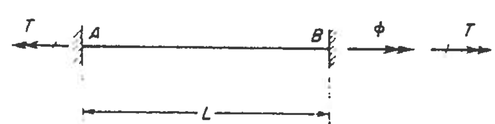
$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$

3



$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$

4



$$T = \frac{GJ\phi}{L}$$

G = shear modulus of elasticity

J = torsion constant