5 Generalized Eigen-Solvers

5.1 Introduction

Eigenvalue problems arise naturally in many engineering applications, such as free structural vibrations, structural dynamics, earthquake engineering, control-structure interactions, etc…

Solution algorithms for both "standard", and "generalized" eigenvalue problems are well documented in the literatures [5.1-5.4, 5.5-5.10]. The focus of this chapter is to explain some efficient algorithms, which take full advantages of sparse technologies, to find the solution of the following generalized eigen-problem:

$$[K]\phi_i = \lambda_i [M]\phi_i \tag{5.1}$$

where:

$[K] \equiv$ n×n stiffness matrix

$[M] \equiv$ n×n mass matrix

$\phi_i \equiv$ eigen-vectors

$\lambda_i \equiv$ eigenvalues

If the mass matrix [M] is equal to an identity matrix, then the generalized eigen-equation (5.1) will become the following standard eigen-problem:

$$[K]\phi_i = \lambda_i \phi_i \tag{5.2}$$

5.2 A simple generalized eigen-example

Considering the following numerical data:

$$[K] = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \tag{5.3}$$

$$[M] = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \tag{5.4}$$

From Eq. (5.1), one has

$$K\phi - \lambda M\phi = 0 \tag{5.5}$$

$$[K - \lambda M]\phi = 0 \tag{5.6}$$

$$\begin{bmatrix} 5 - \frac{5\lambda}{4} & -2 \\ -2 & 2 - \frac{\lambda}{5} \end{bmatrix} \{\phi\} = \{0\} \tag{5.7}$$

The homogeneous system of Eq. (5.7) will have non-trivial solution if we set

$$\det \begin{vmatrix} 5 - \frac{5\lambda}{4} & -2 \\ -2 & 2 - \frac{\lambda}{5} \end{vmatrix} = 0 \tag{5.8}$$

The corresponding 2 roots (= eigenvalues) of the determinant Eq. (5.8) are given as:

$$\lambda = \lambda_1 = 2 \tag{5.9}$$

$$\lambda = \lambda_2 = 12 \tag{5.10}$$

Substituting Eq. (5.9) into Eq. (5.6), one obtains:

$$\begin{bmatrix} 2.5 & -2 \\ -2 & 1.6 \end{bmatrix} \begin{Bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.11}$$

Observing the Eq. (5.11), one recognizes that the 2 equations are NOT independent, and therefore, there is only 1 (= n-1= 2-1) independent equation. Since there are 2 unknowns (= $\phi_1^{(1)}$, and $\phi_2^{(1)}$) and only 1 independent equation, there is infinite number of solutions to Eq. (5.11). A solution can be found by selecting, for example

$$\phi_2^{(1)} = 1 \tag{5.12}$$

Then the other unknown $\phi_1^{(1)}$ can be found from Eq. (5.11) as

$$\phi_1^{(1)} = \frac{4}{5} \tag{5.13}$$

Similarly, substituting Eq. (5.10) into Eq. (5.6), and selecting

$$\phi_2^{(2)} = 2 \tag{5.14}$$

Then, $\phi_1^{(2)}$ can be solved as:

$$\phi_1^{(2)} = -\frac{2}{5} \tag{5.15}$$

Thus, the eigen-matrix $[\Phi]$ can be formed as:

$$[\Phi] = \begin{bmatrix} \phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ 1 & 2 \end{bmatrix} \tag{5.16}$$

Following are 2 properties of the eigen-matrix $[\Phi]$, for the appropriated choice of normalized eigenvectors:

$$[\Phi]^T [M][\Phi] = [I] = \text{Identity matrix} \tag{5.17}$$

$$[\Phi]^T [K][\Phi] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{5.18}$$

5.3 Inverse and forward iteration procedures

An eigenvalue and its associated eigenvector of Eq. (5.1) can be found by the following "inverse iteration" procedure, which is shown in Table 5.1:

Table 5.1 Inverse iteration procedure

Step 1: Guess eigen-pair solution

$$\lambda = 1 \tag{5.19}$$

$$\{\phi\} = \{x_1\} \tag{5.20}$$

Step 2: Substitute Eqs. (5.19-5.20) into Eq. (5.1), hence

$$[K]\{x_1\} \neq (1)[M]\{x_1\} \tag{5.21}$$

The new, better guess $\{x_2\}$ can be found from:

$$[K]\{x_2\} = [M]\{x_1\} \tag{5.22}$$

Step 3: If convergence is not yet achieved, for example

$$\left\| \overrightarrow{x_2} - \overrightarrow{x_1} \right\| > \varepsilon \tag{5.23}$$

then $\{x_1\}$ and $\{x_2\}$ in Eq. (5.22) will be replaced by $\{x_2\}$ and $\{x_3\}$, respectively.

The "forward iteration" procedure is very similar to the inverse iteration. The only difference is the new, better guess $\{x_2\}$ can be found from the following (instead of Eq. 5.22):

$$[K]\{x_1\} = [M]\{x_2\} \tag{5.24}$$

It is, however, more preferable to solve for new, better vector $\{x_2\}$ from Eq. (5.22) rather than from Eq. (5.24), since the former involves with stiffness matrix, which is usually positive definite, and therefore, solution for $\{x_2\}$ can be found easier!

The inverse iteration procedures can often be incorporated with the usage of orthonormality condition, as illustrated in Talbe 5.2.

The following assumptions are made:

(a) The stiffness matrix [K] in Eq. (5.1) is positive definite.
(b) The mass matrix [M] in Eq. (5.1) can be a diagonal mass, with or without zeros on the diagonal. Matrix [M] can also be sparse and/or banded.
(c) If the stiffness matrix [K] is only positive semi-definite, a "shift" should be used prior to the iterations.

Table 5.2 Inverse iterations with orthornormality conditions

Step 1: Guess eigen-pair solution

$$\lambda = 1$$

$$\{\phi\} = \{x_1\}$$

Step 2: For k = 1, 2, 3… until converge, solve for $\bar{x}_{k+1}$ from the following equation

$$[K]\{\bar{x}_{k+1}\} = [M]\{x_k\} \tag{5.25}$$

In order to satisfy the mass orthonormality condition

$$\{x_{k+1}^T\} * [M] * \{x_{k+1}\} = 1 \tag{5.26}$$

We need to orthonormalize the vector $\{\bar{x}_{k+1}\}$ obtained from solving Eq. (5.25), as following:

$$\{x_{k+1}\} = \frac{\{\bar{x}_{k+1}\}}{(\bar{x}_{k+1}^T M \bar{x}_{k+1})^{1/2}} \tag{5.27}$$

The following example is used to illustrate the details involved in the algorithms shown in Table 5.2.

Example: The stiffness and mass matrices [K] and [M] are given as:

$$[K]=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{5.28}$$

$$[M]=\begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \tag{5.29}$$

Initial guesses:

$$\{x_1\}^T=\{1,\quad 1,\quad 1,\quad 1\} \text{ and } \lambda=1 \tag{5.30}$$

Improved guessed vector $\{\bar{x}_2\}$ is solved from:

$$[K]\{\bar{x}_2\}=[M]\{x_1\} \tag{5.31}$$

Thus:

$$\{\bar{x}_2\}^T=\{3,\quad 6,\quad 7,\quad 8\} \tag{5.32}$$

Compute:

$$\{\bar{x}_2\}^T[M]\{\bar{x}_2\}=136 \tag{5.33}$$

Impose the mass of orthonormality condition to obtain the new vector $\{x_2\}$:

$$\{x_2\}=\frac{1}{\sqrt{136}}*\begin{Bmatrix} 3 \\ 6 \\ 7 \\ 8 \end{Bmatrix} \tag{5.34}$$

The above steps are repeated again, to obtain:

$$[K]\{\bar{x}_3\}=[M]\{x_2\} \tag{5.35}$$

Solve for

$$\{\bar{x}_3\}=\left(\frac{1}{\sqrt{136}}\right)*\begin{Bmatrix} 20 \\ 40 \\ 48 \\ 56 \end{Bmatrix} \tag{5.36}$$

$$\{\bar{x}_3\}^T M\{\bar{x}_3\} = \frac{6336}{136} \tag{5.37}$$

$$\{x_3\} = \left(\frac{1}{\sqrt{6336}}\right) * \begin{Bmatrix} 20 \\ 40 \\ 48 \\ 56 \end{Bmatrix} = \begin{Bmatrix} 0.251 \\ 0.503 \\ 0.603 \\ 0.704 \end{Bmatrix} \tag{5.38}$$

The corresponding "exact" eigen-vector is:

$$\phi^{(1)} = \begin{Bmatrix} 0.250 \\ 0.500 \\ 0.606 \\ 0.707 \end{Bmatrix} \tag{5.39}$$

$$[K]\{\phi^{(1)}\} = \lambda_1 [M]\{\phi^{(1)}\} \tag{5.40}$$

It should be emphasized here that it is difficult to assure the convergence to a specific (and arbitrary selected) eigenvalue, and the corresponding eigenvector.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.250 \\ 0.500 \\ 0.602 \\ 0.707 \end{Bmatrix} = \lambda_1 \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \begin{Bmatrix} 0.250 \\ 0.500 \\ 0.602 \\ 0.707 \end{Bmatrix} \tag{5.41}$$

From Eq. (5.41), one has

$$\lambda_1 = 0.148 \tag{5.42}$$

5.4 Shifted eigen-problems

The following example will demonstrate "shifting" procedures can be used to obtain eigen-solutions for the case where the stiffness matrix [K] is singular.

Example: The stiffness and mass matrices [K] and [M] are given as:

$$[K] = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}; [M] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.43}$$

The "shifted" stiffness matrix $[\hat{K}]$ can be computed as

$$\left[\hat{K}\right] = [K] - \rho[M] \tag{5.44}$$

Where ρ is the shifted value.

The "new" eigen-problem is defined as:

$$\left[\hat{K}\right]\{\psi\} = \mu[M]\{\psi\} \tag{5.45A}$$

Substituted Eq. (5.44) into Eq. (5.45) to get

$$(K - \rho M)\psi = \mu M\psi \tag{5.45B}$$

or

$$[K]\{\psi\} = (\rho + \mu)[M]\{\psi\} \tag{5.46}$$

Comparing the "new" eigen-problem of Eq. (5.46) with the "original" eigen-problem shown in Eq. (5.1), one has

$$\lambda = \rho + \mu \tag{5.47}$$

and

$$\{\phi\} = \{\psi\} \tag{5.48}$$

Thus, the eigen-vectors of Eq. (5.1) and Eq. (5.46) are the same, and the eigenvalues of Eqs. (5.1, 5.46) are different only by the shifted value ρ.

For the numerical data shown in Eq. (5.43), one has:

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\{\phi\} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\{\phi\} \tag{5.49}$$

or

$$\begin{bmatrix} 3-2\lambda & -3-\lambda \\ -3-\lambda & 3-2\lambda \end{bmatrix}\{\phi\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.50}$$

For non-trival solution, we require the determinant of the coefficient matrix ($= K - \lambda M$) in Eq. (5.50) to be vanished, thus:

$$\det\left|K - \lambda M\right| = 0 = 3\lambda^2 - 18\lambda = 0 \tag{5.51}$$

or

$$\lambda = \lambda_1 = 0 \tag{5.52}$$

$$\lambda = \lambda_2 = 6 \tag{5.53}$$

Substituting Eq. (5.52) into Eq. (5.50), one obtains the first eigen-vector:

$$\begin{Bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \end{Bmatrix} \tag{5.54}$$

Substituting Eq. (5.53) into Eq. (5.50), one obtains the second eigen-vector

$$\begin{Bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix} \tag{5.55}$$

Now, using the shifted value $\rho = -2$ into Eq. (5.45B), one obtains:

$$\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \{\psi\} = \mu \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \{\psi\} \tag{5.56}$$

The 2 eigen-values from Eq. (5.56) can be obtained as

$$\mu = \mu_1 = 2 \text{ , hence } \lambda_1 = \rho + \mu_1 = -2 + 2 = 0 \tag{5.57}$$

$$\mu = \mu_2 = 8 \text{ , hence } \lambda_2 = \rho + \mu_2 = -2 + 8 = 6 \tag{5.58}$$

Substituting the eigenvalues μ from Eqs. (5.57-5.58) into Eq. (5.56), one obtains the following eigen-vectors

$$\{\psi^{(1)}\} = \begin{Bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \end{Bmatrix} \tag{5.59}$$

$$\{\psi^{(2)}\} = \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix} \tag{5.60}$$

5.5 Transformation Methods

The eigen-vectors $\{\phi^{(i)}\}$obtained from solving the generalized eigen-equation (5.1) have the following propekrties:

$$[\Phi]^T [K][\Phi] = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \end{bmatrix} \tag{5.61}$$

$$[\Phi]^T[M][\Phi] = \begin{bmatrix} \diagdown & & \\ & I & \\ & & \diagup \end{bmatrix} \tag{5.62}$$

Where the eigen-matrix $[\Phi]$ is defined as:

$$[\Phi] = \begin{bmatrix} \phi^{(1)} & \phi^{(2)} & ..... & \phi^{(n)} \end{bmatrix} \tag{5.63}$$

The basic ideas in the transformation methods is to reduce (or transform) the matrices [K] and [M] into diagonal forms, by using successive pre-and post multiplication by matrices $[P_k]^T$, and $[P_k]$, respectively. These ideas are based on the properties given in Eqs. (5.61-5.62). The main ideas behind the transformation method can be summarized in Table 5.3.

Table 5.3 Transformation Methods

$$\text{Let } K_1 = K \text{ and } M_1 = M \tag{5.64}$$

Then:

$$K_2 = P_1^T K_1 P_1 \tag{5.65}$$

$$K_3 = P_2^T K_2 P_2 = P_2^T (P_1^T K_1 P_1) P_2 \tag{5.66}$$

$$\vdots$$

$$K_{k+1} = P_k^T K_k P_k = (P_n^T P_{n-1}^T \cdots P_1^T) K_1 (P_1 P_2 \cdots P_n) \tag{5.67}$$

Similarly:

$$M_2 = P_1^T M_1 P_1 \tag{5.68}$$

$$M_3 = P_2^T M_2 P_2 = P_2^T (P_1^T M_1 P_1) P_2 \tag{5.69}$$

$$\vdots$$

$$M_{k+1} = P_k^T M_k P_k = (P_n^T P_{n-1}^T \cdots P_1^T) M_1 (P_1 P_2 \cdots P_n) \tag{5.70}$$

Comparing Eqs. (5.67, 5.70) and Eqs. (5.61, 5.62), one clearly sees that if matrices P_k are properly selected, then

$$[K_{k+1}] \rightarrow [\lambda] \text{ and } [M_{k+1}] \rightarrow [I] \text{ as } k \rightarrow \infty$$

in which case; with *l* being the last iteration, then the eigen-marix $[\Phi]$can be symbolically represented as

$$\Phi = P_1 P_2 \cdots P_l \tag{5.71}$$

In practice, it is NOT necessary that $[K_{k+1}]$ converges to the diagonal eigen-values matrix $[\lambda]$, and $[M_{k+1}]$ converges to the identity matrix [I]. Rather, it is only required that both $[K_{k+1}]$ and $[M_{k+1}]$ will converge to diagonal matrices, then:

$$[\lambda] = \frac{\text{diagonal } k_r^{(l+1)}}{\text{diagonal } M_r^{(l+1)}} \tag{5.72}$$

and

$$[\Phi] = (P_1 \;\; P_2 \;\; \cdots \;\; P_l) * \frac{1}{\sqrt{\text{diagonal } M_r^{(l+1)}}} \tag{5.73}$$

[A] Jacobi Method For The Standard Eigen-Problems

In this section, the following standard eigen-problem

$$[K]\{\phi\} = \lambda\{\phi\} \tag{5.74}$$

will be solved by the Jacobi method. Eq. (5.74) is a special case of the generalized eigen-equation (5.1), where [M] is set to be the identity matrix.

From Eqs. (5.70, 5.62), and realizing [M] = [I], one has:

$$M_{k+1} = P_k^T [I] P_k = [I]$$

or

$$[P_k]^T [P_k] = [I] \tag{5.75}$$

Thus, the transformation methods (such as Jacobi method) requires the selected matrix $[P_k]$ to have the properties as indicated in Eq. (5.75). The matrix $[P_k]$, therefore, should have the following form:

$$P_k = \begin{bmatrix} 1 & & & \vdots & & & & & \vdots & & \\ & 1 & & \vdots & & & & & \vdots & & \\ & & 1 & \vdots & & & & & \vdots & & \\ & & & \cos\theta & & & & & -\sin\theta & & \\ & & & \vdots & 1 & & & & \vdots & & \\ & & & \vdots & & 1 & & & \vdots & & \\ & & & \vdots & & & 1 & & \vdots & & \\ & & & \vdots & & & & 1 & \vdots & & \\ \cdots & \cdots & \cdots & \sin\theta & \cdots & \cdots & \cdots & \cdots & \cos\theta & \cdots & \cdots \\ & & & \vdots & & & & & \vdots & 1 & \\ & & & \vdots & & & & & \vdots & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \rightarrow i^{th}\text{row} \\ \\ \rightarrow j^{th}\text{row} \\ \\ \\ \end{matrix} \quad (5.76)$$

$$\begin{matrix} & & & i^{th}\text{column} & & & & & j^{th}\text{column} & & \end{matrix}$$

Where θ is selected such that the element (i, j) of $[K_{k+1}]$ at $(k+1)^{th}$ iteration will become zero (since the main objective of the Jacobi method is to transform the original matrix [K] into a diagonal matrix, where all the off-diagonal terms $K_{i,j}$ will become zeros).

Considering the following tripple products:

$$K_{k+1} = P_k^T K_k P_K, \text{ or} \quad (5.77)$$

$$\begin{bmatrix} a & b & c & d & e & f \\ b & \bar{g} & h & i & \bar{j} & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & \bar{j} & n & q & \bar{s} & t \\ f & k & o & r & t & u \end{bmatrix} = \quad (5.78)$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cos\theta & & & \sin\theta & \\ \cdot & & 1 & & & \\ \cdot & & & 1 & & \\ \cdot & -\sin\theta & & & \cos\theta & \\ \cdot & & & & & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \\ b & g & h & i & j & k \\ c & h & l & m & n & o \\ d & i & m & p & q & r \\ e & j & n & q & s & t \\ f & k & o & r & t & u \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cos\theta & & & -\sin\theta & \\ \cdot & & 1 & & & \\ \cdot & & & 1 & & \\ \cdot & \sin\theta & & & \cos\theta & \\ \cdot & & & & & 1 \end{bmatrix}$$

Assuming that i = 2th and j = 5th locations, and observing Eq. (5.78), one clearly sees that $[K_{k+1}]$ and $[K_k]$ are only different at locations (2, 2), (2, 5), (5, 2) and (5, 5), all remaining terms of matrices $[K_{k+1}]$ and $[K_k]$ are unchanged. Thus, at the $(k+1)^{th}$ iteration, one only needs to consider the following portions of the triple product:

$$K_{k+1} = P_k^T K_k P_k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} K_{ii}^{(k)} & K_{ij}^{(k)} \\ K_{ji}^{(k)} & K_{jj}^{(k)} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5.79}$$

where:

$$[K_{k+1}] = \begin{bmatrix} K_{ii}^{(k+1)} & K_{ij}^{(k+1)} \\ K_{ji}^{(k+1)} & K_{jj}^{(k+1)} \end{bmatrix}$$

The objective to select the matrix P_k (at the k^{th} iteration) is to make sure that in the next iteration (or the $(k+1)^{th}$ iteration), the off-diagonal terms of $[K_{k+1}]$ will be driven to zeros. Hence, setting the off-diagonal terms $K_{ij}^{(k+1)} = K_{ji}^{(k+1)} = 0$, and equating with the corresponding right-hand-side terms of Eq. (5.78), one can solve for the unknown θ, from either of the following formulas:

$$\tan(2\theta) = \frac{2K_{ij}^{(k)}}{K_{ii}^{(k)} - K_{jj}^{(k)}}, \text{for } K_{ii}^{(k)} \neq K_{jj}^{(k)} \tag{5.80}$$

$$\text{or } \theta = \frac{\pi}{4}, \text{for } K_{ii}^{(k)} = K_{jj}^{(k)} \tag{5.81}$$

If "ℓ" is the last iteration, then:

$$[K_{\ell+1}] \approx \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \text{diagonal eigenvalues matrix} \tag{5.82}$$

and we said convergence to a tolerance ξ has been achieved, provided that the following conditions are met:

(a) Diagonal terms do NOT change much, or

$$\frac{\left|K_{ii}^{(\ell+1)} - K_{ii}^{(\ell)}\right|}{K_{ii}^{(\ell+1)}} \le 10^{-\xi} \quad ; \quad i = 1, 2, \cdots, n \tag{5.83}$$

(b) Off-diagonal terms approach zeros, or

$$\left[\frac{K_{ij}^{(\ell+1)\ 2}}{K_{ii}^{(\ell+1)} K_{jj}^{(\ell+1)}}\right]^{\frac{1}{2}} \le 10^{-\xi} \quad ; \quad i < j \tag{5.84}$$

The step-by-step Jacobi method for the solution of the standard eigenvalue problem can be given as followings:

Step 1: At the s^{th} sweep, a threshold for a near zero tolerance value is defined as 10^{-2s}.

Step 2: For all $K_{i,j}$ terms (with i<j, and i, j = 1, 2, ..., n = size of matrix [K]), compute the coupling factor according to the left-hand-side of Eq. (5.84), and apply the transformation (see Eq. 5.79) if the factor is larger than the current threshold.

Step 3: Using Eq. (5.83) to check for convergence. If convergence is achieved, then the process is stopped, or else, returning to step 1 for another sweep.

Example

$$\text{Given } [K] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \equiv [K_1]$$

For sweep 1 we have as a threshhold 10^{-2}. We therefore obtain the following results. For i = 1, j = 2, applying Eq.(5.80), one gets:

$$\cos\theta = 0.7497\,;\ \sin\theta = 0.6618$$

and thus, from Eq (5.76), one has:

$$P_1 = \begin{bmatrix} 0.7497 & -0.6618 & 0 & 0 \\ 0.6618 & 0.7497 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_1^T K P_1 = \begin{bmatrix} 1.169 & 0 & -1.898 & 0.6618 \\ 0 & 9.531 & -3.661 & 0.7497 \\ -1.898 & -3.661 & 6 & -4 \\ 0.6618 & 0.7497 & -4 & 5 \end{bmatrix} = [K_2]$$

For i = 1, j = 3:

$$\cos\theta = 0.9398\,;\ \sin\theta = 0.3416$$

$$P_2 = \begin{bmatrix} 0.9398 & 0 & -0.3416 & 0 \\ 0 & 1 & 0 & 0 \\ 0.3416 & 0 & 0.9398 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_2^T P_1^T K P_1 P_2 = \begin{bmatrix} 0.7792 & -1.2506 & 0 & -0.7444 \\ -1.2506 & 9.5314 & -3.4402 & 0.7497 \\ 0 & -3.4402 & 6.6891 & -3.9853 \\ -0.7444 & 0.7497 & -3.9853 & 5. \end{bmatrix} = [K_3]$$

$$P_1 P_2 = \begin{bmatrix} 0.7046 & -0.6618 & -0.2561 & 0 \\ 0.6220 & 0.7497 & -0.2261 & 0 \\ 0.3416 & 0 & 0.9398 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For i = 1, j = 4:

$$\cos\theta = 0.9857\,;\ \sin\theta = 0.1687$$

$$P_3 = \begin{bmatrix} 0.9857 & 0 & 0 & -0.1687 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.1687 & 0 & 0 & 0.9857 \end{bmatrix}$$

$$P_3^T P_2^T P_1^T K P_1 P_2 P_3 = \begin{bmatrix} 0.6518 & -1.106 & -0.6725 & 0 \\ -1.106 & 9.531 & -3.440 & 0.9499 \\ -0.6725 & -3.440 & 6.690 & -3.928 \\ 0 & 0.9499 & -3.928 & 5.127 \end{bmatrix} = [K_4]$$

$$P_1 P_2 P_3 = \begin{bmatrix} 0.6945 & -0.6618 & -0.2561 & -0.1189 \\ 0.6131 & 0.7497 & -0.2261 & -0.1050 \\ 0.3367 & 0 & 0.9398 & -0.0576 \\ 0.1687 & 0 & 0 & 0.9857 \end{bmatrix}$$

For i = 2, j = 3:

$$\cos\theta = 0.8312\,;\ \sin\theta = -0.5560$$

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8312 & 0.5560 & 0 \\ 0 & -0.5560 & 0.8312 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_4^T P_3^T P_2^T P_1^T K P_1 P_2 P_3 P_4 = \begin{bmatrix} 0.6518 & 0.5453 & -1.174 & 0 \\ -0.5453 & 11.83 & 0 & 2.974 \\ -1.174 & 0 & 4.388 & -2.737 \\ 0 & 2.974 & -2.737 & 5.127 \end{bmatrix} = [K_5]$$

$$P_1 P_2 P_3 P_4 = \begin{bmatrix} 0.6945 & -0.4077 & -0.5808 & -0.1189 \\ 0.6131 & 0.7488 & 0.2289 & -0.1050 \\ 0.3367 & -0.5226 & 0.7812 & -0.0576 \\ 0.1682 & 0 & 0 & 0.9857 \end{bmatrix}$$

To complete the first sweep, we zero element (3,4), using

$$\cos\theta = 0.7335\,;\ \sin\theta = -0.6797$$

$$P_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.7335 & 0.6797 \\ 0 & 0 & -0.6797 & 0.7335 \end{bmatrix}$$

and hence the approximation obtained for Λ and Φ are

$$\Lambda = P_6^T \ldots P_1^T K P_1 \ldots P_6$$

i.e.,

$$\Lambda = \begin{bmatrix} 0.6518 & -0.5098 & 0.9926 & -0.6560 \\ 0.5098 & 12.96 & -0.7124 & -0.6601 \\ -0.9926 & -0.7124 & 6.7596 & 0 \\ -0.6560 & -0.6602 & 0 & 1.6272 \end{bmatrix}$$

and

$$\Phi \doteq P_1 \ldots P_6$$

i.e.,

$$\Phi \doteq \begin{bmatrix} 0.6945 & -0.4233 & -0.4488 & -0.3702 \\ 0.6131 & 0.6628 & 0.4152 & -0.1113 \\ 0.3367 & -0.5090 & 0.4835 & 0.6275 \\ 0.1687 & 0.3498 & -0.6264 & 0.6759 \end{bmatrix}$$

And after the third sweep we have

$$\Lambda \doteq \begin{bmatrix} 0.1459 & & & \\ & 13.09 & & \\ & & 6.854 & \\ & & & 1.910 \end{bmatrix}$$

$$\Phi \doteq \begin{bmatrix} 0.3717 & -0.3717 & -0.6015 & -0.6015 \\ 0.6015 & 0.6015 & 0.3717 & -0.3717 \\ 0.6015 & -0.6015 & 0.3717 & 0.3717 \\ 0.3717 & 0.3717 & -0.6015 & 0.6015 \end{bmatrix}$$

The approximation for Λ is diagonal to the precision given and we can use

$$\lambda_1 \doteq 0.1459; \qquad \phi_1 \doteq \begin{bmatrix} 0.3717 \\ 0.6015 \\ 0.6015 \\ 0.3717 \end{bmatrix}$$

$$\lambda_2 \doteq 1.910; \qquad \phi_2 \doteq \begin{bmatrix} -0.6015 \\ -0.3717 \\ 0.3717 \\ 0.6015 \end{bmatrix}$$

$$\lambda_3 \doteq 6.854; \qquad \phi_3 \doteq \begin{bmatrix} -0.6015 \\ 0.3717 \\ 0.3717 \\ -0.6015 \end{bmatrix}$$

$$\lambda_4 \doteq 13.09; \qquad \phi_4 \doteq \begin{bmatrix} -0.3717 \\ 0.6015 \\ -0.6015 \\ 0.3717 \end{bmatrix}$$

[B] Generalized Jacobi Method For Generalized Eigen-Problems

For the generalized eigen-problem as described by Eq. (5.1), we wish to diagonalize both the stiffness and mass matrices [K], and [M], respectively. Thus, the transformation matrix $[P_k]$ will be selected as:

$$P_k = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & \theta_2 & & & i^{th} \text{ row} \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & \theta_1 & & & 1 & & j^{th} \text{ row} \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix} \tag{5.85}$$

The new, updated matrices [K] and [M] in the next iteration can be computed from the current iteration as following:

$$[K_{k+1}] = P_k^T K_k P_k \tag{5.86}$$

or

$$\begin{bmatrix} K_{ii}^{(k+1)} & K_{ij}^{(k+1)} \\ K_{ji}^{(k+1)} & K_{jj}^{(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 \\ \theta_2 & 1 \end{bmatrix} \begin{bmatrix} K_{ii}^{(k)} & K_{ij}^{(k)} \\ K_{ji}^{(k)} & K_{jj}^{(k)} \end{bmatrix} \begin{bmatrix} 1 & \theta_2 \\ \theta_1 & 1 \end{bmatrix} \tag{5.87}$$

$$= \begin{bmatrix} K_{ii}^{(k)} + \theta_1 K_{ji}^{(k)} & K_{ij}^{(k)} + \theta_1 K_{jj}^{(k)} \\ K_{ji}^{(k)} + \theta_2 K_{ii}^{(k)} & K_{jj}^{(k)} + \theta_2 K_{ij}^{(k)} \end{bmatrix} \begin{bmatrix} 1 & \theta_2 \\ \theta_1 & 1 \end{bmatrix} \tag{5.88}$$

or

$$\begin{bmatrix} K_{ii}^{(k+1)} & 0 \\ 0 & K_{jj}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \times & \theta_2\{K_{ii}^{(k)} + \theta_1 K_{ji}^{(k)}\} + \{K_{ij}^{(k)} + \theta_1 K_{jj}^{(k)}\} \\ \times & \times \end{bmatrix} \tag{5.89}$$

The off-diagonal terms on both sides of Eq. (5.89) are equated to each other, thus:

$$K_{ij}^{(k+1)} = 0 = \theta_2 K_{ii}^{(k)} + \{1 + \theta_1\theta_2\} K_{ij}^{(k)} + \theta_1 K_{jj}^{(k)} \tag{5.90}$$

Similarly, one has:

$$[M_{k+1}] = P_k^T M_k P_k \tag{5.91}$$

$$M_{ij}^{(k+1)} = 0 = \theta_2 M_{ii}^{(k)} + \{1 + \theta_1\theta_2\} M_{ij}^{(k)} + \theta_1 M_{jj}^{(k)} \tag{5.92}$$

The two unknowns θ_1 and θ_2 can be found by solving Eqs. (5.90, 5.92) simultaneously [5.5]

$$\theta_1 = \frac{-G_1^{(k)}}{G_4^{(k)}} \quad \text{and} \quad \theta_2 = \frac{G_2^{(k)}}{G_4^{(k)}} \tag{5.93}$$

where:

$$G_1^{(k)} \equiv K_{ii}^{(k)} M_{ij}^{(k)} - M_{ii}^{(k)} K_{ij}^{(k)} \tag{5.94}$$

$$G_2^{(k)} \equiv K_{jj}^{(k)} M_{ij}^{(k)} - M_{jj}^{(k)} K_{ij}^{(k)} \tag{5.95}$$

$$G_3^{(k)} \equiv K_{ii}^{(k)} M_{jj}^{(k)} - K_{jj}^{(k)} M_{ii}^{(k)} \tag{5.96}$$

$$G_4^{(k)} = \frac{G_3^{(k)}}{2} + \text{sign}\{G_3^{(k)}\} * \sqrt{\left\{\frac{G_3^{(k)}}{2}\right\}^2 + G_1^{(k)} * G_2^{(k)}} \tag{5.97}$$

Remarks

(a) Eq. (5.93) for θ_1 and θ_2 have been developed for the case [M] is a positive definite, full or banded matrix, and it can be proved that $G_4^{(k)}$ is always non-zero.

(b) The generalized Jacobi procedure can also be adopted for the case [M] is a diagonal matrix, with or without zero diagonal elements.

(c) Assuming "*l*" is the last iteration, then convergence is achieved if

$$\frac{\left|\lambda_i^{(l+1)} - \lambda_i^{(l)}\right|}{\lambda_i^{(l+1)}} \le 10^{-\xi}, \text{ for i = 1, 2, ..., n} \tag{5.98}$$

where:

$$\lambda_i^{(l)} \equiv \frac{K_{ii}^{(l)}}{M_{ii}^{(l)}} \text{ and } \lambda_i^{(l+1)} \equiv \frac{K_{ii}^{(l+1)}}{M_{ii}^{(l+1)}} \tag{5.99}$$

and

$$\left[\frac{\left\{K_{ij}^{(l+1)}\right\}^2}{K_{ii}^{(l+1)} K_{jj}^{(l+1)}}\right]^{1/2} \le 10^{-\xi} \tag{5.100}$$

$$\left[\frac{\left\{M_{ij}^{(l+1)}\right\}^2}{M_{ii}^{(l+1)} M_{jj}^{(l+1)}}\right]^{1/2} \le 10^{-\xi} \tag{5.101}$$

for all (i, j) with i<j

5.6 Subspace iteration method[5.5]

Subspace iteration method is an effective algorithm to find the "few" lowest eigen-pairs of fairly large generalized eigen-problem. Both inverse iteration, and the generalized Jacobi iteration methods have been incorporated into the subspace iteration algorithm. The main steps involved in this method can be described as following:

Assuming the first "m" eigen-pair solution for Eq. (5.1) are sought. One computes:

$$L = \text{Minimum } (2*m,\ m+8) \tag{5.102}$$

and $L \le n$ (= size of matrix [K])

"Guess" the first L eigen-vectors matrix $[X_1]_{n\times L}$

For k = 1, 2, 3, ... until convergence is achieved.

- Solve $\overline{X}_{k+1}$ from:

$$[K]\left[\overline{X}_{k+1}\right] = [M][X_k] \tag{5.103}$$

- Find the "reduced" stiffness and mass matrices from:

$$\left[K^R_{k+1}\right]_{L\times L} = \left[\overline{X}^T_{k+1}\right]_{L\times n}[K]_{n\times n}\left[\overline{X}_{k+1}\right]_{n\times L} \tag{5.104}$$

$$\left[M^R_{k+1}\right]_{L\times L} = \left[\overline{X}^T_{k+1}\right][M]\left[\overline{X}_{k+1}\right] \tag{5.105}$$

- Solve the "reduced" eigen-problem

$$\left[K^R_{k+1}\right]_{L\times L}\left[Q_{k+1}\right]_{L\times L} = [\wedge_{k+1}]_{L\times L}\left[M^R_{k+1}\right]_{L\times L}\left[Q_{k+1}\right]_{L\times L} \tag{5.106}$$

- Find the improved eigen-vectors

$$[X_{k+1}]_{n\times L} = \left[\overline{X}_{k+1}\right]_{n\times L}[Q_{k+1}]_{L\times L} \tag{5.107}$$

Then

$[\wedge_{k+1}] \to [\lambda] \equiv$ eigenvalues and $[X_{k+1}] \to [\Phi] \equiv$ eigenvectors (see Eq.5.1)

as $k \to \infty$

Remarks:

(a) Inverse iteration method is employed in Eq. (5.103)

(b) Generalized Jacobi iteration method can be used to solve the reduced eigen-equation (5.106).

(c) The initial guess eigen-vector $[X_1]$ should contain independent columns. The simplest guess for $[X_1]$ is:

$$[X_1]_{n\times L} = \left[e^{(1)} \quad e^{(2)} \quad \cdots \quad e^{(L)}\right] \tag{5.108}$$

where

$$e^{(i)} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{n\times 1} \rightarrow i^{th}\text{row} = \text{unit vector} \tag{5.109}$$

Example:

The numerical data for the stiffness and mass matrix [K] and [M] are given as:

$$[K] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad [M] = \begin{bmatrix} \frac{1}{2} & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix} \tag{5.110}$$

Assuming the initial guess for eigen-vectors is given by

$$[X_1] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \tag{5.111}$$

$[\overline{X}_2]$ can be solved from Eq. (5.103):

$$[\overline{X}_2] = \left(\frac{1}{4}\right)\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \tag{5.112}$$

Reduced stiffness and mass matrices can be found from Eqs. (5.104-5.105):

$$[K_2^R] = \overline{X}_2{}^T K \overline{X}_2 = \left(\frac{1}{4}\right)\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \tag{5.113}$$

$$[M_2^R] = \overline{X}_2{}^T M \overline{X}_2 = \left(\frac{1}{16}\right)\begin{bmatrix} 9 & 7 \\ 7 & 9 \end{bmatrix} \tag{5.114}$$

Solution of the reduced eigen-problem

$$\left[K_2^R\right][Q_2]=[\Lambda_2]\left[M_2^R\right][Q_2]$$

can be solved "iteratively" (by using Generalized Jacobi method), or can also be solved "directly", and "exactly" since the system involved with only 2×2 matrices.

Thus:

$$[\Lambda_2]=\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}=\text{eigenvalues of the reduced system} \tag{5.115}$$

$$[Q_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & 2\\ \frac{1}{\sqrt{2}} & -2\end{bmatrix}=\text{eigen-vectors of the reduced system} \tag{5.116}$$

The new, improved eigen-vectors can be obtained from Eq. (5.107):

$$[X_2]=\left[\bar{X}_2\right][Q_2]=\begin{bmatrix}\frac{1}{\sqrt{2}} & -1\\ \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 1\end{bmatrix} \tag{5.117}$$

In this particular example, we obtained the "exact" eigen-values (see Eq. 5.115), and "exact" eigen-vectors (see Eq. 5.117) in 1 interation (of the subspace iteration loop)! This quick convergence is due to the fact that the starting iteration vectors $[X_1]$ (see Eq. 5.111) span the subspace defined the "exact" eigen-vectors $\phi^{(1)}$ and $\phi^{(2)}$.

Proof (for the above paragraph)

The first 2 "exact" eigen-vectors from a 3×3 system given by Eq. (5.110) are:

$$\phi^{(1)}=\begin{Bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 1/\sqrt{2}\end{Bmatrix} \text{ and } \phi^{(2)}=\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix} \tag{5.118}$$

$$\phi^{(1)} = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} = a_1 \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} + a_2 \begin{Bmatrix} 2 \\ 1 \\ 0 \end{Bmatrix} \tag{5.119}$$

(1^{st} column of $[X_1]$; 2^{nd} column of $[X_1]$)

where $a_1 = 1/\left(2\sqrt{2}\right) = a_2$

and

$$\phi^{(2)} = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = a_3 \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} + a_4 \begin{Bmatrix} 2 \\ 1 \\ 0 \end{Bmatrix} \tag{5.120}$$

where $a_3 = \frac{1}{2}$; $a_4 = \frac{-1}{2}$

In other words, the exact first 2 eigen-vectors $\phi^{(1)}$ and $\phi^{(2)}$ can be expressed as linear combinations of the columns of the starting iteration vectors $[X_1]$. Thus, subspace iteration algorithm converges in just 1 iteration.

5.7 Lanczos eigen-solution algorithms

For large-scale system, if one has interests in obtaining the "first few" eigen-pair solutions for the generalized eigen-equation (5.1), then subspace iteration algorithms can be employed. However, if the number of requested eigen-pair solutions increase, then the computational cost involved in subspace iteration algorithms will increase very quickly. Thus, in this case, Lanczos eigen-solution algorithms is highly recommended (due to its highly computational efficiency)