Simple Head-Tail Matching Game for Teaching Linear Programming Simplex Algorithms With Real-World Applications.

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Abstract. Linear Programming (LP) is a very popular/important topic, with very broad/real-world Engineering/Economic, and Social Science applications. In addition to Operation Research, LP course has been offered in most (if not all) engineering curriculum in the USA. Teaching LP topic, however, may not be an easy task, especially at the undergraduate, and/or even at the high-school levels. Through simple Head-Tail game strategies, and coupling with graphical methods, the authors hope that the formulation and optimal solution for LP problem can be easily understood even by high-school students.

1. SIMPLE HEAD-TAIL GAME STRATEGIES

In the proposed "Matching Head/Tail" game, suppose both players (S for Susan and V for Victoria) use different strategies such as varying the relative frequency for observing Heads or Tails when tossing the coins. For example, player S might adopt her strategy to be 0.7 probability of selecting Heads and 0.3 of selecting Tails. Similarly, player V might want to play Heads and Tails using a different probability: 0.6 for Heads and 0.4 for Tails. This game with different strategies can be conveniently represented in matrix notation as shown in Table 1.

Table 1: A 2x2 Payoff Matrix (unequal frequency for Heads and Tails)

			Player V	
			H	T
			c = 0.6	d = 0.4
Player S	H	a = 0.7	1	-2
	T	b = 0.3	-3	4

In the above table, an entry 1 means S wins $1 for the case S has Heads and V also has Heads. Similarly, an entry -2 means S loses $2 for the case S has Heads and V has Tails. Using the above turning wheel strategies (shown in Figure 1), the probability of having Heads and Tails for V are six-tenth and four-tenth, respectively. One can easily show (see the following 3-step derivation) that in this game, S actually loses to V an average of $0.2 per play in the long run.

Step 1. S plays Heads seven-tenth of the time, while V plays Heads six-tenth of the time and Tails four-tenths of the time respectively. Thus, S's average winnings for this scenario can be computed as:

S1 = (0.7) [(0.6) (1) + (0.4) (-2)] = -$0.14 per play

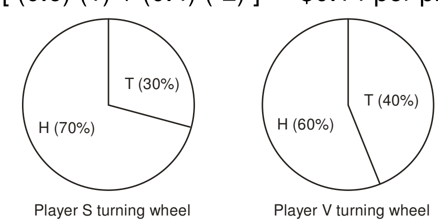

Figure 1: Turning wheel strategy for both players

Step 2. S plays Tails three-tenth of the time, while V plays Heads six-tenth of the time and Tails four-tenths of the time. Thus, S's average winnings for these occasions are:

S2 = (0.3) [(0.6) (-3) + (0.4) (4)] = -$0.06 per play

Step 3. Adding the above 2 amounts, S will win an average of:

S_{total} = S1 + S2 = (-0.14) + (-0.06) = -0.20

The negative result implies that in fact, in the long run, S actually loses to V an average of $0.2 per play. For this given 2x2 payoff matrix, the best strategies for S and V are (a=0.7, b=0.3), and (c=0.6, d=0.4), respectively. The value for this game is, therefore, $0.2. This value represents maximum gain for V, and at the same time, minimum loss for S. Clearly, this game scenario favors V! A classroom discussion, and computer implementation (using powerful animated FLASH software [1]) of the above game strategies can potentially draw students to broaden their thinking.

2. GAME-BASED LINEAR PROGRAMMING (LP) OPTIMIZATION FORMULATION

Game strategy for S is (a,b) , such that

$$a \geq 0\,;\, b \geq 0 \tag{1}$$

$$a + b = 1 \tag{2}$$

In the above Eqs. (1-2), "a" and "b" represents the desired/selected probability for S to observe HEAD and TAIL, respectively.

S expects to win over V's two "pure strategies" (1,0), and (0,1) respectively by the following amounts:

$$a[(1)(1)+(0)(-2)]+b[(1)(-3)+(0)(4)] = 1a - 3b \tag{3}$$

$$a[(0)(1)+(1)(-2)]+b[(0)(-3)+(1)(4)] = -2a + 4b \tag{4}$$

Define:

s = minimum (a-3b, -2a+4b) (5)

Notes: s = minimum (amount to win) = worse case for S

S does NOT know exact strategies played by V. However, S knows for sure that V's strategies (c,d) must be some things between the 2 pure strategies!

$$(1)a + (-3)b \geq s \tag{6}$$

$$(-2)a + (4)b \geq s \tag{7}$$

It is valid to assume that s to have positive value, since if this is NOT true, then one can add a suitable positive constant k to each term of 2x2 matrix (shown earlier in Table 1), so that Eqs. (6,7) become:

$$(1+k)a + (-3+k)b \geq s \tag{8}$$

$$(-2+k)a + (4+k)b \geq s \tag{9}$$

or

$$1a - 3b + k(a+b) \geq s \tag{10}$$

$$-2a + 4b + k(a+b) \geq s \tag{11}$$

Utilizing Eq. (2), Eqs. (10-11) becomes:

$$1a - 3b + k \geq s \tag{12}$$

$$-2a + 4b + k \geq s \tag{13}$$

where k can be selected as:

k = absolute value of {smallest entry of the given matrix}. (14)

For this particular data:

k = abs{smallest entry of (1,-3,-2,4)}

k = abs{-3} = 3 (15)

Substituting Eq. (15) into Eqs. (8-9), then:

$$4a + 0b \geq s \tag{16}$$

$$1a + 7b \geq s \tag{17}$$

Due to Eq. (1), the left-hand-side (LHS) of Eqs. (16-17) are positive, and the required Eq. (5) becomes:

$$s = \min(4a + 0b, 1a + 7b) \geq 0 \tag{18}$$

Thus, by adding a suitable positive constant k (if necessary, see Eq. 14), it would simply increase the expected payoff by the amount k (comparing Eq. 5 with Eqs. 12-13), and it would "NOT" change the optimal strategies.

Dividing Eqs. (1, 2, 16, 17) by s , one obtains:

$$\frac{a}{s} \geq 0, \frac{b}{s} \geq 0 \tag{19}$$

$$\frac{a}{s} + \frac{b}{s} = \frac{1}{s} \tag{20}$$

$$4\left(\frac{a}{s}\right) + 0\left(\frac{b}{s}\right) \geq 1 \tag{21}$$

$$1\left(\frac{a}{s}\right) + 7\left(\frac{b}{s}\right) \geq 1 \tag{22}$$

Since "S ≡ Susan" would like to <u>maximize</u> the payoff (or winning amount) s , she would like to <u>minimize</u> $\left(\frac{1}{s}\right)$.

Define:

$$t \equiv \frac{1}{s};\, u \equiv \frac{a}{s};\, and\ v \equiv \frac{b}{s} \tag{23}$$

Then, Eqs. (19-22) become the following "Dual" problem:

Minimize $t = u + v$ (24)

such that

$$(4)u+(0)v \geq 1 \quad (25)$$
$$(1)u+(7)v \geq 1 \quad (26)$$
$$u \geq 0\,;\, v \geq 0 \quad (27)$$

3. DUALITY LINEAR PROGRAMMING (LP) PROBLEMS

Now, consider "V's ≡ Victoria's" strategies (c,d), such that

$$c \geq 0\,;\, d \geq 0 \quad (28)$$
$$c+d=1 \quad (29)$$

V expect to lose over S's two pure strategies (1,0), and (0,1), respectively by the following amounts:

$$c[(1)(1)+(0)(-3)]+d[(1)(-2)+(0)(4)]=1c-2d \quad (30)$$
$$c[(0)(1)+(1)(-3)]+d[(0)(-2)+(1)(4)]=-3c+4d \quad (31)$$

Since not all entries of the given 2x2 matrix are positive, a positive constant $k=3$ can be selected (see Eq. 14) and added to Eqs. (30-31) to give:

$$(1+k)c+(-2+k)d=(4)c+(1)d \quad (32)$$
$$(-3+k)c+(4+k)d=(0)c+(7)d \quad (33)$$

Define:

w = maximum (4c+1d, 0c+7d) (34)
= worse case for **V**

Then:

$$(4)c+(1)d \leq w \quad (35)$$
$$(0)c+(7)d \leq w \quad (36)$$

Dividing Eqs. (28, 29, 35, 36) by w, one gets:

$$\frac{c}{w} \geq 0\,;\, \frac{c}{w} \geq 0 \quad (37)$$
$$\frac{c}{w}+\frac{d}{w}=\frac{1}{w} \quad (38)$$
$$4\left(\frac{c}{w}\right)+1\left(\frac{d}{w}\right) \leq 1 \quad (39)$$
$$0\left(\frac{c}{w}\right)+7\left(\frac{d}{w}\right) \leq 1 \quad (40)$$

Since, V would like to minimize (the loss) w, or maximize $\frac{1}{w}$, one defines:

$$x \equiv \frac{1}{w}\,;\; y \equiv \frac{c}{w}\,;\; and\;\; z \equiv \frac{d}{w} \quad (41)$$

Then, Eqs. (37-40) become the following "Primal" Problem:

Maximize $x=y+z$ (42)
such that

$$(4)y+(1)z \leq 1 \quad (43)$$
$$(0)y+(7)z \leq 1 \quad (44)$$
$$y \geq 0\,;\, z \geq 0 \quad (45)$$

The linear programming (LP) problems, shown in Eqs. (24-27): the "Dual" problem, and Eqs. (42-45): the "Primal" problem, respectively, will have the following properties [2, 5]:

1. If optimum solution exists for either LP problem, then so does the other LP problem, and

   $$\left|t^* \equiv t_{\min imum}\right| = \left|x^* \equiv x_{\max imum}\right|$$

2. If the optimum solution of one of the above LP problems is unbounded, then the other LP problem has no feasible solution!

4. GRAPHICAL SOLUTION FOR THE LP "PRIMAL" PROBLEM

The LP primal problem, shown in Eqs. (42-45), can be presented in a graphical form as shown in Figure 2

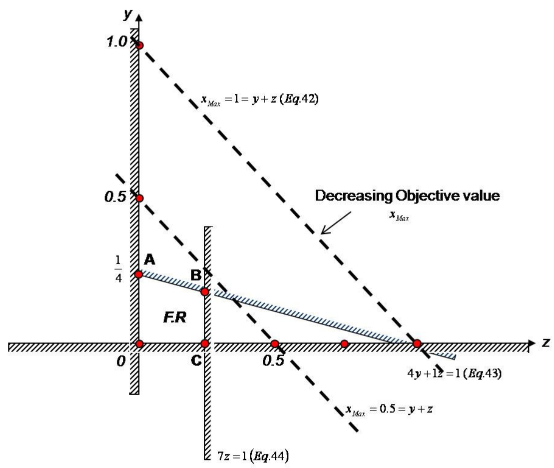

Figure 2: Optimum Point is at Point B

The coordinates at ‘optimal point” B can be obtained from the two constraint Eqs. (43-44):

$$4y+1z=1 \Rightarrow 4y+\frac{1}{7}=1 \Rightarrow y^*=\frac{3}{14}$$

$$7z=1 \Rightarrow z^*=\frac{1}{7}$$

The “optimum solution” can be computed from Eq. (42) as:

$$x^*_{Max}=y^*+z^*=\frac{3}{14}+\frac{1}{7}=\frac{5}{14}$$

$$\frac{5}{14}=x^*_{Max}$$

5. STANDARD FORM OF LP PROBLEMS [2]

The standard form of LP problems should satisfy the following requirements:

a) Solving an “Maximization” problem.
b) All “inequality” constraints must be converted to “equality” constraints.
c) All right hand side (RHS) values of the constraints must be ≥ 0 (or non negative).
d) All “original” and newly created ‘slack, surplus, artificial” variables must be ≥ 0.

As a quick example, Tables 2-4 can be used to explain the process to convert the original LP problem (see Table 2) into the (final) standard LP problem (see Tables 3-4).

Table 2: Original Problem

$$F_{\min} = x+5y$$
$$2x-3y \leq 8$$
$$x+2y \geq -4$$
$$x+y \geq 6$$
$$x \geq 0$$
$$y = \textit{No restrictions in signs.}$$

Table 3: Standard LP problem

$$\overline{F}_{\max} = -(x+5y) = -x-5y$$
$$2x-3y+S_1 = 8$$
$$-(x+2y) \leq 4 \Rightarrow -x-2y+S_2 = 4$$
$$x+y \quad -S_3 = 6$$
$$x \geq 0$$
$$y = y_1 - y_2,\ \textit{where}\ y_1 \geq 0\ \&\ y_2 \geq 0$$

Table 4. Standard LP problem (final form)

$$F_{\max} = -x-5(y_1-y_2)$$
$$2x-3(y_1-y_2)+S_1 = 8$$
$$-x-2(y_1-y_2) \quad +S_2 = 4$$
$$x+(y_1-y_2) \quad -S_3+A_1 = 6$$
$$x \geq 0;\ S_1 \geq 0;\ S_2 \geq 0;\ S_3 \geq 0$$
$$y_1 \geq 0;\ y_2 \geq 0;\ A_1 \geq 0$$

Notes: x, y_1, y_2 = Original variables
S_1, S_2, S_3 = Slack/Surplus variables
A_1 = Artificial variable.

6. SIMPLEX SOLUTION FOR THE LP “PRIMAL” PROBLEM

In the standard form, the LP primal problem, shown in Eqs. (42-45), can be expressed as:

$$4y+1z+S_1 = 1 \quad (46)$$
$$0y+7z+ \quad S_2 = 1 \quad (47)$$
$$y+z = x_{Max} \quad (48)$$
$$y, z, S_1, S_2 \geq 0 \quad (49)$$

For the above standard LP problem, one has 2 (equality) constraints (see Eqs. 46-47) and 4 unknown variables (basic variables S_1, S_2 and non-basic variables y, z). Non-basic variables, by definitions, will have “Zero” numerical values. Basic variables, therefore, are the ones which have “canonical forms” and have their numerical values to be equal to the RHS of constraint equations (46-47).

The basic/popular Simplex Algorithms [2,5] are based on the following main ideas:

1. In each iteration, we have to decide which non-basic variable will be selected to ENTER into the basic variable group?, and
2. Which basic variable will be KICKED OUT from the basic variable group?
3. The objective function (see Eq. 48) should always be expressed in terms of "non-basic variables" to satisfy the "canonical form" requirement.
4. The iterative process will be stopped if no further improvements can be done.

. To answer the above question 1, we need to look at the objective function (see Eq. 48). The selected (non-basic) variable to ENTER the basic variable group should be the one associated with the largest positive coefficient, since this choice will help the objective function $(x_{\max})$ the most!

In this example, since both variables y and z have the same coefficient value (=1), we can arbitrarily select variable z to ENTER the basic variable group. This implies the variable y to be remained in the "non-basic" group, and hence:

$$y = 0 \tag{50}$$

. To answer the above question 2, the current basic variables S_1 *and* S_2 can be solved from the constraint equations (46-47):

$$S_1 = 1 - 4y - z \geq 0 \tag{51}$$

$$S_2 = 1 - 0y - 7z \geq 0 \tag{52}$$

Eqs. (51-52) imply:

$$z \leq 1 \tag{51A}$$

$$z \leq \frac{1}{7} \tag{52A}$$

In the above 2 equations, Eq. (52A) will control since if this equation (or constraint requirement) is satisfied, then the other constraint requirement (see Eq. 51A) is also automatically satisfied!

The largest value z may have, therefore, is

$$z = \frac{1}{7} \tag{52B}$$

Substituting Eq. (52 B) into Eqs. (51-52), one gets:

$$S_1 = 1 - \left(z = \frac{1}{7} \right) = \frac{6}{7} \tag{53}$$

$$S_2 = 1 - 7\left(z = \frac{1}{7} \right) = 0 \tag{54}$$

Comparing Eqs. (53-54), since $S_2 = 0$, hence the basic variable S_2 should be one to be kicked out from the basic group (or entering into the non-basic group)!

The above iterative process is repeated (see the following Simplex tables) until optimum solution is obtained. Large-scale LP revised Simplex matrix factorization algorithms can also be implemented in parallel computer environments [3] to fully exploit parallel processing capability offered by most modern computers.

Basic	y	z	S_1	S_2	RHS = b	$\frac{b}{a}$
S_1	4	1	1	0	1	1
S_2	0	(7)	0	1	1	($\frac{1}{7}$) → out
	1	1 ↑ in	0	0	x_{Max}	
S_1	(4)	0	1	$-\frac{1}{7}$	$\frac{6}{7}$	($\frac{3}{14}$) → out
z	0	1	0	$\frac{1}{7}$	$\frac{1}{7}$	∞
	1 ↑ in	0	0	$-\frac{1}{7}$	$-\frac{1}{7}+x_{Max}$	
y	1	0	$\frac{1}{4}$	$-\frac{1}{28}$	$\frac{3}{14}$	
z	0	1	0	$\frac{1}{7}$	$\frac{1}{7}$	
	0	0	$-\frac{1}{4}$	$-\frac{3}{28}$	$-\frac{5}{14}+x_{Max}$	

Hence, optimum point is $(y^*, z^*) = \left(\frac{3}{14}, \frac{1}{7} \right)$

and the optimum solution is $x^*_{Max} = \frac{5}{14}$, which has the same value as earlier obtained in Section 4 (using the Graphical Method) !

7. BIG_M SIMPLEX SOLUTION METHOD [2,5]

Artificial variables are "NOT" required in the above LP problem, since it only involves "<" type constraints. However, for those LP problems that have ">, or =" type constraints, then artificial variables need be introduced (in order to have proper canonical form, and to have the starting SIMPLEX table !). for convenience, let's introduce artificial variables A_1, and A_2 into the 2 constraint Eqs. (46-47), and the objective function Eq. (48), respectively. Then, the following Big_M SIMPLEX LP problem can be formulated:

$$4y+1z+S_1 \qquad +A_1 \qquad =1 \tag{55}$$

$$0y+7z \qquad +S_2 \qquad +A_2 \;=1 \tag{56}$$

and

$$(y+z)-M(A_1+A_2) \qquad = x\_\max \tag{57}$$

In Eq. (57), the "new" objective function $x\_\max$ consists of the "original" objective function, and augmented by adding a "penalty function", where M = any "big" number, say M = 1,000,000. This arbitrarily large value of M will ensure the SIMPLEX algorithm to force all artificial variables to eventually become "zero" (meaning $A_1 = A_2 = 0$), and Eq. (57) will become the original objective function Eq. (48)!

Since the objective function has to be expressed in terms of "non-basic variables" only (in order to preserve the canonical form), the basic/artificial variables A_1, and A_2 can be expressed in term of "non-basic" variables by solving the 2 constraint Eqs. (55-56), as following:

$$A_1 = 1-4y-z-S_1 \tag{58}$$

$$A_2 = 1 \qquad -7z \qquad -S_2 \tag{59}$$

Substituting Eqs. (58-59) into Eq. (57), one obtains:

$$(1+4M)y+(1+8M)z+(M)S_1+(M)S_2 = x\_\max+2M \tag{60}$$

Based on the above Eqs. (55-56, 60), the familiar SIMPLEX procedures can be generated/computed, as shown in the following tables:

Basic	y	z	S_1	S_2	A_1	A_2	RHS = b	$\frac{b}{a}$	
A_1	4	1	1	0	1	0	1	1	
A_2	0	(7)	0	1	0	1	1	$\frac{1}{7}$	→ out
	1+4M	1+8M ↑ in	M	M	0	0	$x_{\max}+2M$		
A_1	(4)	0	1	$-\frac{1}{7}$	1	$-\frac{1}{7}$	$\frac{6}{7}$	$\frac{3}{14}$	→ out
z	0	1	0	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{1}{7}$	∞	
	1+4M ↑ in	0	M	$-\frac{1}{7}-\frac{M}{7}$	0	$-\frac{1}{7}-\frac{8M}{7}$	$x_{\max}-\frac{1}{7}+\frac{6M}{7}$		
y	1	0	$\frac{1}{4}$	$-\frac{1}{28}$	$\frac{1}{4}$	$-\frac{1}{28}$	$\frac{3}{14}$		
z	0	1	0	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{1}{7}$		
	0	0	$-\frac{1}{4}$	$-\frac{3}{28}$	$-\frac{1}{4}-M$	$-\frac{3}{28}-M$	$x_{\max}-\frac{5}{14}$		

$$y^* = \frac{3}{14}$$

$$z^* = \frac{1}{7}$$

$$S_1^* = 0 = S_2^* = A_1^* = A_2^* = 0$$

$$0 = x_{\max} - \frac{5}{14} \Rightarrow x_{\max} = \frac{5}{14}$$

8. REAL-WORLD "RACIAL DESEGREGATION OF SCHOOL/BUS SYSTEMS" APPLICATION [2,4]

Since the landmark Supreme Court decision in 1954 invalidating school segregation, the problem of racial desegregation of school systems has received a great deal of attention. In urban communities, where residential patterns produce de facto segregation of schools, many school administrations have adopted an official policy of eliminating such segregation by busing students (or else face the loss of federal funds).

With the above considerations in mind, the problem chosen is that of using available mass transportation most effectively to achieve a given ethnic mix in each school in a community.

The following data are given, which can be obtained from census data, or resulting from policy decisions.

Table 5: Distributions of Ethnic Groups, School Districts and Student Populations

	District			Total by	
Ethnic Group	**1**	**2**	**3**	**Ethnic Group**	**%**
A	900	100	0	1,000	40
B	200	600	100	900	36
C	100	100	400	600	24
Totals	1200	800	500	2,500	100

Table 6: Schools' Capacities

School No.	Maximum Capacity (No. of Students)
I	500
II	800
III	700
IV	700

Table 7: Allowable Percentage of Ethnic Groups at any school

	Allowable Composition	
Ethnic Group	**Minimum %**	**Maximum %**
A	36	46
B	32	40
C	20	28

Table 8: Travel Times (minutes) between any district to any school

District	**I**	**II**	**III**	**IV**
1	12	13	21	31
2	18	13	12	22
3	34	30	25	17

Using the above given data, and the Simplex algorithm explained in Section 6, undergraduate students will be able to formulate the LP problem, and find the "optimum" solution which

1. Places each student in a school.
2. Achieves an ethnic composition within the given ranges for each school.
3. Minimizes the total daily student transportation time (or any other objective that one might consider pertinent).

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