More Examples ....

where

$$B(v, w) = \int_0^L b \frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\,dx$$

$$l(v) = \int_0^L vf\,dx + \left(\frac{dv}{dx}\right)\Big|_{x=L} M_0 \tag{2.56c}$$

The quadratic form, commonly known as the *total potential energy* of the beam, is obtained using (2.56*c*) and (2.43*b*):

$$I(w) = \int_0^L \left[\frac{b}{2}\left(\frac{d^2w}{dx^2}\right)^2 - wf\right]dx - \left(\frac{dw}{dx}\right)\Big|_{x=L} M_0 \tag{2.57}$$

Note that for the fourth-order equation, the essential boundary conditions involve not only the dependent variable but also its first derivative. As pointed out earlier, at any boundary point, only one of the two boundary conditions (essential or natural) can be specified. For example, if the transverse deflection is specified at a boundary point then one cannot specify the shear force V at the same point, and vice versa. Similar comments apply to the slope dw/dx and the bending moment M. Note that in the present case, w and dw/dx are the primary variables, and V and M are the secondary variables.

Geometric b.c. Natural b.c.

The next example is concerned with a second-order differential equation governing conductive and convective heat transfer in two dimensions. It should be noted that the boundary condition for a convective boundary contains both primary and secondary variables.

Example 2.3. Consider steady heat conduction in a two-dimensional domain Ω, enclosed by lines AB, BC, CD, DE, EF, FG, GH, and HA (see Fig. 2.2). The governing equation is

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = q_0 \quad \text{in } \Omega \tag{2.58}$$

where q_0 is the uniform heat generation, k is the conductivity of the isotropic material of the domain, and T is the temperature. We wish to construct the weak form of the

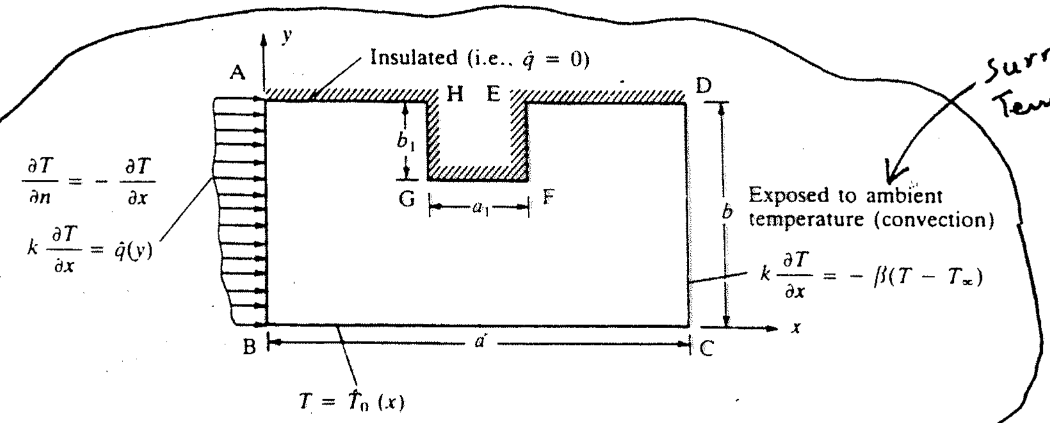

FIGURE 2.2
Conduction and convection heat transfer in two-dimensional domains.

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where w denotes the
$G = \partial T/\partial y$ in (2.22*b*)]

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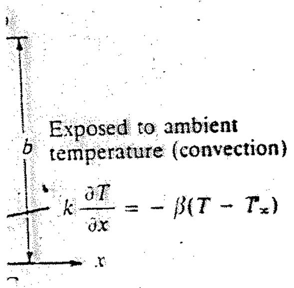

equation. Equation (2.58), known as the Poisson equation, arises in many fields of engineering (see Table 8.1).

Proceeding as described earlier, we have

step 1 $\longrightarrow$
$$0 = \int_\Omega w\left[-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - q_0\right] dx\, dy$$

where w denotes the weight function. Using (2.22) [with $G = \partial T/\partial x$ in (2.22a) and $G = \partial T/\partial y$ in (2.22b)], we obtain

similar to integ. by parts
$$0 = \int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) - wq_0\right] dx\, dy - \oint_\Gamma wk\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right) ds \qquad (2.59)$$

The reader should verify the last step [i.e. the application of (2.22)]. From the boundary expression, it follows that the secondary variable of the problem is of the form

$$k\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right) = k\frac{\partial T}{\partial n} \equiv q_n$$

and the primary variable is T. The secondary variable q_n denotes the total flux across (i.e., along the normal to) the boundary. In general, q_n is composed of fluxes due to conduction, convection, and radiation.

The boundary Γ of the domain consists of several line segments, and they are subject to different types of boundary conditions (see Fig. 2.2):

on $\Gamma_1 = AB$ ($n_x = -1, n_y = 0$):	specified heat flux, $\hat{q}(y)$	
on $\Gamma_2 = BC$ ($n_x = 0, n_y = -1$):	specified temperature, $\hat{T}_0(x)$ $\equiv x$, on Γ_{CD}	
on $\Gamma_3 = CD$ ($n_x = 1, n_y = 0$):	convective boundary with ambient temperature T_∞ and film coefficient β: $k\,\partial T/\partial n + \beta(T - T_\infty) = 0$	(2.60)
on $\Gamma_4 = DEFGHA$:	insulated boundary, $\partial T/\partial n = 0$	

Using the boundary information, the boundary integral in (2.59) can be simplified as follows (note that $w = 0$ on Γ_2):

$$\oint_\Gamma w\left(k\frac{\partial T}{\partial n}\right) ds = \int_{\Gamma_1} wq_n\, ds + \int_{\Gamma_2} 0\left(k\frac{\partial T}{\partial n}\right) ds$$
$$- \int_{\Gamma_3} w[\beta(T - T_\infty)]\, ds + \int_{\Gamma_4} w0\, ds$$
$$= -\int_0^b w(0, y)\hat{q}(y)\, dy - \beta\int_0^b w(a, y)[T(a, y) - T_\infty]\, dy \qquad (2.61)$$

Substituting (2.61) into (2.59), we obtain the weak form

$$0 = \int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) - wq_0\right] dx\, dy + \int_0^b w(0, y)\hat{q}(y)\, dy$$
$$+ \beta\int_0^b w(a, y)[T(a, y) - T_\infty]\, dy \qquad (2.62)$$

Collecting terms involving both w and T into $B(\cdot, \cdot)$, and those involving only w into $l(\cdot)$, we can write (2.62) in the form

$$B(w, T) = l(w) \qquad (2.63a)$$

where

$$B(w, T) = \int_\Omega k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) dx\, dy + \beta \int_0^b w(a, y)T(a, y)\, dy$$

$$l(w) = \int_\Omega w q_0\, dx\, dy - \int_0^b w(0, y)\hat{q}(y)\, dy + \beta \int_0^b w(a, y)T_\infty\, dy \qquad (2.63b)$$

The quadratic functional is given by

$$I(T) = \frac{k}{2}\int_\Omega \left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] dx\, dy - \int_\Omega T q_0\, dx\, dy + \int_0^b T(0, y)\hat{q}(y)\, dy + \beta \int_0^b \tfrac{1}{2}[T^2(a, y) - 2T(a, y)T_\infty]\, dy \qquad (2.63c)$$

Note that the boundary integrals in this example are defined along the y and x axes, respectively. This is because the boundaries are parallel to either the x or the y axis.

2.4 VARIATIONAL METHODS OF APPROXIMATION

2.4.1 Introduction

Our objective in this section is to study the variational methods of approximation. These include the Rayleigh–Ritz, Galerkin, Petrov–Galerkin, least-squares, and collocation methods. In all these, we seek an approximate solution in the form of a linear combination of suitable approximation functions ϕ_j and undetermined parameters c_j: $\sum_j c_j\phi_j$. The parameters c_j are determined such that the approximate solution satisfies the weighted-integral form or weak form of the governing equation or minimizes the quadratic functional associated with the equation studied. Various methods differ from each other in the choice of weight function w and approximation functions ϕ_j.

The primary objective of this section is to present a number of classical variational methods. The finite element method makes use of variational methods to formulate the discrete equations over an element. As we shall see in Chapters 3–14, the choice of the approximation functions in the finite element methods is different from that in the classical variational methods.

2.4.2 The Rayleigh–Ritz Method

In the Rayleigh–Ritz method, the coefficients c_j of the approximation are determined using the weak form of the problem, and the choice of weight functions is restricted to the approximation functions, $w = \phi_j$. Recall that the weak form contains both the governing differential equation and the natural boundary conditions of the problem, and it places less stringent continuity requirements on the approximate solution than the original differential equation or its weighted-integral form. The method is described below for a linear variational problem.

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Consider the variational problem of finding the solution u such that

$$B(w, u) = l(w) \tag{2.64}$$

for all sufficiently differentiable functions w that satisfy the homogeneous form of any specified essential boundary conditions on u. When the functional B is bilinear and symmetric and l is linear, the problem in (2.64) is equivalent to minimization of the quadratic functional

$$I(u) = \tfrac{1}{2}B(u, u) - l(u) \tag{2.65}$$

In the Rayleigh–Ritz method, we seek an approximate solution to (2.64) in the form of a finite series

$$\boxed{u_N = \sum_{j=1}^{N} c_j\phi_j + \phi_0} \tag{2.66}$$

where the constants c_j, called the *Ritz coefficients*, are chosen such that (2.64) holds for $w = \phi_i$ $(i = 1, 2, \ldots, N)$; i.e., (2.64) holds for N different choices of w, so that N independent algebraic equations in c_j are obtained. The requirements on ϕ_j and ϕ_0 will be discussed shortly. The ith algebraic equation is obtained by substituting ϕ_i for w:

$$B\left(\phi_i, \sum_{j=1}^{N} c_j\phi_j + \phi_0\right) = l(\phi_i) \quad (i = 1, 2, \ldots, N)$$

If B is bilinear, the summation and constants c_j can be taken outside the operator. We have

$$\sum_{j=1}^{N} B(\phi_i, \phi_j)c_j = l(\phi_i) - B(\phi_i, \phi_0) \tag{2.67a}$$

or

$$\sum_{j=1}^{N} B_{ij}c_j = F_i, \qquad B_{ij} = B(\phi_i, \phi_j), \qquad F_i = l(\phi_i) - B(\phi_i, \phi_0) \tag{2.67b}$$

which represents the ith algebraic equation in a system of N linear algebraic equations in N constants c_j. The columns (and rows) of the matrix coefficients $B_{ij} = B(\phi_i, \phi_j)$ must be linearly independent in order that the coefficient matrix in (2.67) can be inverted.

For symmetric bilinear forms, the Rayleigh–Ritz method can also be viewed as one that seeks a solution of the form in (2.66) in which the parameters are determined by minimizing the quadratic functional corresponding to the symmetric bilinear form, that is, the functional $I(u)$ in (2.65). After substituting u_N from (2.66) for u into (2.65) and integrating, the functional $I(u)$ becomes an ordinary (quadratic) function of the parameters $c_1, c_2, \ldots$. Then the necessary condition for the minimization of $I(c_1, c_2, \ldots, c_N)$ is that its partial derivatives with respect to each of the parameters be zero:

$$\frac{\partial I}{\partial c_1}=0, \quad \frac{\partial I}{\partial c_2}=0, \quad \ldots, \quad \frac{\partial I}{\partial c_N}=0 \tag{2.68}$$

Thus there are N linear algebraic equations in N unknowns, c_j ($j = 1, 2, \ldots, N$). These equations are exactly the same as those in (2.67) for all problems for which the variational problem (2.64) is equivalent to $\delta I = 0$. Of course, when $B(\cdot, \cdot)$ is not symmetric, we do not have a quadratic functional. In other words, (2.67) is more general than (2.68), and they are the same when $B(\cdot, \cdot)$ is bilinear and symmetric. In most problems of interest in the present study, we shall have a symmetric bilinear form.

Returning to the Rayleigh–Ritz approximation u_N in (2.66), we note that u_N must satisfy the specified essential boundary conditions of the problem; any specified natural boundary conditions are already included in the variational problem (2.64). The particular form of u_N in (2.66) facilitates satisfaction of specified boundary conditions. If we were to use the form

$u_N = \sum_j c_j \phi_j + \phi_0$

$$u_N = \sum_{j=1}^{N} c_j \phi_j(x)$$

then it would not be easy to satisfy nonhomogeneous boundary conditions. For example, suppose that u_N is required to satisfy the condition $u_N(x_0) = u_0$ at a boundary point $x = x_0$:

$$\sum_{j=1}^{N} c_j \phi_j(x_0) = u_0$$

Since c_j are unknown parameters to be determined, it is not easy to choose $\phi_j(x)$ such that this relation holds. If $u_0 = 0$ then any ϕ_j such that $\phi_j(x_0) = 0$ would meet the requirement. By writing the approximate solution u_N in the form (2.66), a sum of homogeneous and nonhomogeneous parts, the nonhomogeneous essential boundary conditions can be satisfied by ϕ_0, $\phi_0(x_0) = u_0$, and ϕ_j are required to satisfy the homogeneous form of the same boundary condition, $\phi_j(x_0) = 0$. In this way, u_N satisfies the specified boundary conditions:

Homo. b.c. (Essential b.c.)

$$u_N(x_0) = \sum_{j=1}^{N} c_j \phi_j(x_0) + \phi_0(x_0)$$

Non.Homo. b.c. (Essential b.c.)

$$= 0 + u_0$$

If all specified essential boundary conditions are homogeneous (i.e., the specified value u_0 is zero) then ϕ_0 is taken to be zero and ϕ_j must still satisfy the same conditions, $\phi_j(x_0) = 0$. Since ϕ_i satisfy the homogeneous essential boundary conditions, the choice $w = \phi_i$ is consistent with the requirements of a weight function. The approximation functions ϕ_i satisfy the following zero:

(2.68)

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conditions:

1. (*a*) ϕ_i should be such that $B(\phi_i, \phi_j)$ is well defined and nonzero [i.e., sufficiently differentiable as required by the bilinear form $B(\cdot, \cdot)$].
   (*b*) ϕ_i must satisfy at least the homogeneous form of the essential boundary conditions of the problem. (2.69)
2. For any N, the set $\{\phi_i\}_{i=1}^{N}$ along with the columns (and rows) of $B(\phi_i, \phi_j)$ must be linearly independent.
3. $\{\phi_i\}$ must be complete. For example, when ϕ_i are algebraic polynomials, completeness requires that the set $\{\phi_i\}$ should contain all terms of the lowest order admissible, and up to the highest order desired.

The only role that ϕ_0 plays is to satisfy the specified nonhomogeneous essential boundary conditions of the problem. Any low-order function that satisfies the specified essential boundary conditions should be used. If all specified essential boundary conditions are homogeneous then $\phi_0 = 0$ and

$$F_i = l(\phi_i) - B(\phi_i, \phi_0) = l(\phi_i) \tag{2.70}$$

Next, we consider a few examples of the application of the Rayleigh–Ritz method.

Example 2.4. Consider the differential equation [cf. Example 2.1, with $a = c = 1$]

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for } 0 < x < 1 \tag{2.71}$$

We consider two sets of boundary conditions:

$$\textbf{set 1:} \quad u(0) = 0, \quad u(1) = 0 \tag{2.72a}$$

$$\textbf{set 2:} \quad u(0) = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1 \tag{2.72b}$$

Set 1. The bilinear functional and the linear functional are [see (2.47*c*)]

$$B(w, u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu\right) dx, \qquad l(w) = -\int_0^1 wx^2\, dx \tag{2.73}$$

Since both boundary conditions $[u(0) = u(1) = 0]$ are of the essential type, we must select ϕ_i in the N-parameter Ritz approximation to satisfy the conditions $\phi_i(0) = \phi_i(1) = 0$. We choose the following functions: $\phi_0 = 0$ and

$$\phi_1 = x(1-x), \quad \phi_2 = x^2(1-x), \quad \ldots, \quad \phi_N = x^N(1-x) \tag{2.74}$$

It should be pointed out that if one selects, for example, the functions $\phi_1 = x^2(1-x)$, $\phi_2 = x^3(1-x)$, etc. [not including $x(1-x)$], requirement 3 in the conditions (2.69) is violated, because the set cannot be used to generate the linear term x if the exact solution contains it. As a rule, one must start with the lowest-order admissible function and include all admissible, higher-order functions up to the desired degree.

Note: Basic $\phi_i = (x-0)(x-1)[a_1 + a_2 x + a_3 x^2 \ldots]$
Let $w = \phi_1, \phi_2 \ldots$

The N-parameter Rayleigh–Ritz solution for the problem is of the form

$$u_N = c_1\phi_1 + c_2\phi_2 + \ldots + c_N\phi_N = \sum_{j=1}^{N} c_j\phi_j \tag{2.75}$$

Substituting this into the variational problem $B(w, u) = l(w)$, we obtain

$$\int_0^1 \left[\frac{d\phi_i}{dx}\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}\right) - \phi_i\left(\sum_{j=1}^{N} c_j\phi_j\right)\right] dx = -\int_0^1 \phi_i x^2\, dx$$

$$\sum_{j=1}^{N} c_j \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j\right) dx = -\int_0^1 \phi_i x^2\, dx$$

or

$$\sum_{j=1}^{N} c_j B(\phi_i, \phi_j) = l(\phi_i) \tag{2.76a}$$

where the coefficients $B(\phi_i, \phi_j)$ and $l(\phi_i)$ are defined by

$$B(\phi_i, \phi_j) = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j\right) dx, \qquad l(\phi_i) = -\int_0^1 x^2\phi_i\, dx \tag{2.76b}$$

The same result can be obtained using (2.65) [instead of (2.64)]. We have

Alternative Approach

$$I(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{du}{dx}\right)^2 - u^2 + 2x^2u\right] dx$$

Substituting for $u \approx u_N$ from (2.75) into the above functional, we obtain

$$I(c_j) = \frac{1}{2}\int_0^1 \left[\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}\right)^2 - \left(\sum_{j=1}^{N} c_j\phi_j\right)^2 + 2x^2\left(\sum_{j=1}^{N} c_j\phi_j\right)\right] dx \tag{2.77}$$

The necessary conditions for the minimization of I, which is a quadratic function of the variables $c_1, c_2, \ldots, c_N$, are

$$\frac{\partial I}{\partial c_i} = 0 = \int_0^1 \left[\frac{d\phi_i}{dx}\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}\right) - \phi_i\left(\sum_{j=1}^{N} c_j\phi_j\right) + \phi_i x^2\right] dx$$

$$= \sum_{j=1}^{N} B_{ij}c_j - F_i$$

where

$$B_{ij} = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j\right) dx, \qquad F_i = -\int_0^1 x^2\phi_i\, dx$$

which are the same as those in (2.76). Equations (2.76a, b) hold for any choice of admissible approximation functions ϕ_i.

For the choice of approximation functions in (2.74), the matrix coefficients $B_{ij} \equiv B(\phi_i, \phi_j)$ and vector coefficients $F_i \equiv l(\phi_i) - B(\phi_i, \phi_0) = l(\phi_i)$ can be computed using

$$\phi_i = x^i(1-x) = x^i - x^{i+1}$$

$$\frac{d\phi_i}{dx} = ix^{i-1} - (i+1)x^i$$

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$$B_{ij} = \int_0^1 \{[ix^{i-1} - (i+1)x^i][jx^{j-1} - (j+1)x^j] - (x^i - x^{i+1})(x^j - x^{j+1})\}\, dx$$

$$= \frac{2ij}{(i+j)[(i+j)^2 - 1]} - \frac{2}{(i+j+1)(i+j+2)(i+j+3)} \tag{2.78a}$$

$$F_i = -\int_0^1 x^2(x^i - x^{i+1})\, dx = -\frac{1}{(3+i)(4+i)} \tag{2.78b}$$

Equation (2.76) can be written in matrix form as

$$[B]\{c\} = \{F\} \tag{2.79}$$

For example, when $N = 2$, (2.79) becomes

$$\frac{1}{420}\begin{bmatrix} 126 & 63 \\ 63 & 52 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = -\frac{1}{60}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

and the use of Cramer's rule to solve the equations gives

$$c_1 = -\tfrac{10}{123} = -0.0813, \qquad c_2 = -\tfrac{21}{123} = -0.1707.$$

The two-parameter Rayleigh–Ritz solution is given by

$$u_2 = c_1\phi_1 + c_2\phi_2 = (-\tfrac{10}{123})(x - x^2) + (-\tfrac{21}{123})(x^2 - x^3)$$
$$= -\tfrac{1}{123}(10x + 11x^2 - 21x^3)$$

The exact solution of (2.71) and (2.72a) is given by

$$u(x) = \frac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2 \tag{2.80}$$

The values of the Ritz coefficients for various values of N can be obtained by solving (2.79). A comparison of the Rayleigh–Ritz solution (2.75) with the exact solution (2.80) is presented in Table 2.1 and Fig. 2.3.

Set 2. For the second set of boundary conditions (2.72b), the bilinear form is the same as that given in (2.73) and (2.76b). The linear form is given by ($\phi_0 = 0$)

From Eq. (2.47 C)

$$l(w) = -\int_0^1 wx^2\, dx + w(1) \tag{2.81a}$$

and we therefore have

$$F_i = -\int_0^1 x^2\phi_i\, dx + \boxed{\phi_i(1)} \tag{2.81b}$$

In this case, the ϕ_i should be selected to satisfy the condition $\phi_i(0) = 0$, because the only EBC is at $x = 0$. The following choice of ϕ_i meets the requirements:

$$\phi_i = x^i \tag{2.82}$$

The coefficients B_{ij} and F_i can be computed using (2.82) in (2.76b) and (2.81b) respectively:

$$B_{ij} = \int_0^1 (ijx^{i+j-2} - x^{i+j})\, dx = \frac{ij}{i+j-1} - \frac{1}{i+j+1}$$
$$F_i = -\int_0^1 x^{i+2}\, dx + \boxed{1} = -\frac{1}{i+3} + 1 \tag{2.83}$$

$$[x^i]_{@x=1} = 1$$

TABLE 2.1
Comparison of the Rayleigh–Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for } 0 < x < 1; \qquad u(0) = u(1) = 0$$

Ritz coefficients†		x	Rayleigh–Ritz solution, $-10u$: $N=1$	$N=2$	$N=3$	Exact solution
$N=1$:		0.0	0.0	0.0	0.0	0.0
	$c_1 = -0.1667$	0.1	0.1500	0.0885	0.0954	0.0955
$N=2$:		0.2	0.2667	0.1847	0.1890	0.1890
	$c_1 = -0.0813$	0.3	0.3500	0.2783	0.2766	0.2764
	$c_2 = -0.1707$	0.4	0.4000	0.3590	0.3520	0.3518
$N=3$:		0.5	0.4167	0.4167	0.4076	0.4076
	$c_1 = -0.0952$	0.6	0.4000	0.4410	0.4340	0.4342
	$c_2 = -0.1005$	0.7	0.3500	0.4217	0.4200	0.4203
	$c_3 = -0.0702$	0.8	0.2667	0.3486	0.3529	0.3530
		0.9	0.1500	0.2115	0.2183	0.2182
		1.0	0.0	0.0	0.0	0.0

† The four-parameter Rayleigh–Ritz solution coincides with the exact solution up to four decimal places.

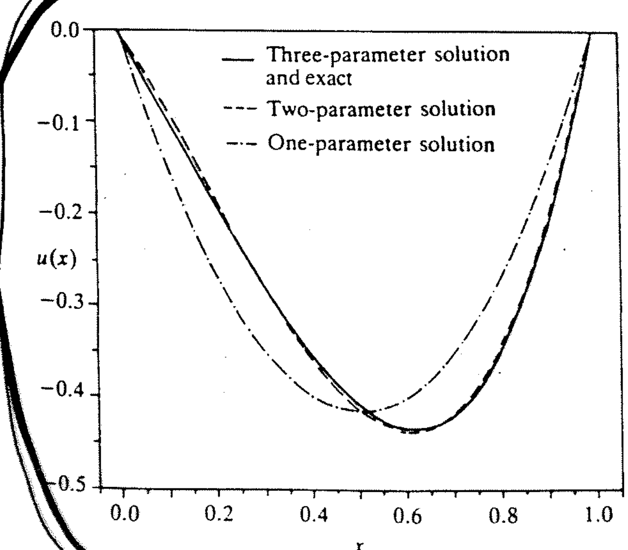


FIGURE 2.3
Comparison of the Rayleigh–Ritz solution with the exact solution of (2.71) and (2.72*a*). The three-parameter solution and the exact solution do not differ on the scale of the plot.

TABLE 2.2
Comparison of the R
equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{fc}$$

Ritz coefficients†	
$N=1$:	
	$c_1 = 1.1250$
$N=2$:	
	$c_1 = 1.2950$
	$c_2 = -0.15108$
$N=3$:	
	$c_1 = 1.2831$
	$c_2 = -0.11424$
	$c_3 = -0.02462$

† The four-parameter Rayleigh-
decimal places.

The exact solution

A comparison of the R
Table 2.2.

Example 2.5. Consider
beam under a uniform
M_0 using Euler–Bernou
this theory are

$$w(0) = \left(\frac{dw}{dx}\right)\Big|_{x=}$$

The variational form
Example 2.2, and is giv
We now constru
(2.56), $B(v, w) = l(v)$,

$$B(v, w) =$$

TABLE 2.2
Comparison of the Rayleigh–Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for } 0<x<1; \qquad u(0)=0, \qquad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1$$

Ritz coefficients†		x	Rayleigh–Ritz solution, u: $N=1$	$N=2$	$N=3$	Exact solution
$N=1$:		0.0	0.0	0.0	0.0	0.0
	$c_1 = 1.1250$	0.1	0.1125	0.1280	0.1271	0.1262
$N=2$:		0.2	0.2250	0.2530	0.2519	0.2513
	$c_1 = 1.2950$	0.3	0.3375	0.3749	0.3740	0.3742
	$c_2 = -0.15108$	0.4	0.4500	0.4938	0.4934	0.4944
$N=3$:		0.5	0.5625	0.6097	0.6099	0.6112
	$c_1 = 1.2831$	0.6	0.6750	0.7226	0.7234	0.7244
	$c_2 = -0.11424$	0.7	0.7875	0.8325	0.8337	0.8340
	$c_3 = -0.02462$	0.8	0.9000	0.9393	0.9407	0.9402
		0.9	1.0125	1.0431	1.0443	1.0433
		1.0	1.1250	1.1439	1.1442	1.1442

† The four-parameter Rayleigh–Ritz solution coincides with the exact solution up to four decimal places.

The exact solution in the present case is given by

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2 \tag{2.84}$$

A comparison of the Rayleigh–Ritz solution with the exact solution is presented in Table 2.2.

Example 2.5. Consider the problem of finding the transverse deflection of a cantilever beam under a uniform transverse load of intensity f_0 per unit length and end moment M_0 using Euler–Bernoulli beam theory (see Example 2.2). The governing equations of this theory are

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - f_0 = 0 \quad \text{for } \begin{cases} 0<x<L \\ EI>0 \end{cases} \tag{2.85}$$

$$w(0) = \left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \qquad \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = M_0, \qquad \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x=L} = 0 \tag{2.86}$$

The variational form of (2.85) (which includes the specified NBC) was derived in Example 2.2, and is given by (2.56).

We now construct an N-parameter Ritz solution using the variational form, (2.56), $B(v, w) = l(v)$, where

$$B(v, w) = \int_0^L EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}\,dx, \qquad l(v) = \int_0^L f_0 v\,dx + \left(M_0\frac{dv}{dx}\right)\bigg|_{x=L} \tag{2.87}$$

Note that the specified EBC, $w(0) = 0$ and $(dw/dx)|_{x=0}$ are homogeneous. Therefore, $\phi_0 = 0$. We select algebraic approximation functions ϕ_i that satisfy the continuity conditions and boundary conditions $\phi_i(0) = \phi_i'(0) = 0$. The lowest-order algebraic function that meets these conditions is $\phi_1 = x^2$. The next function in the sequence is $\phi_2 = x^3$. Thus we have

$$\phi_1 = x^2, \quad \phi_2 = x^3, \quad \ldots, \quad \phi_N = x^{N+1}$$

The N-parameter Rayleigh–Ritz approximation is

$$w_N(x) = \sum_{j=1}^{N} c_j\phi_j, \quad \phi_j = x^{j+1} \tag{2.88}$$

Substituting (2.88) for w and $v = \phi_i$ into (2.87), we obtain

$$\begin{aligned} B_{ij} &= \int_0^L EI(i+1)ix^{i-1}(j+1)jx^{j-1}\,dx = \frac{EIij(i+1)(j+1)L^{i+j-1}}{i+j-1} \\ F_i &= \frac{f_0(L)^{i+2}}{i+2} + M_0(i+1)L^i \end{aligned} \tag{2.89}$$

For $N = 2$ (i.e., the two-parameter solution), we have

$$\begin{aligned} EI(4Lc_1 + 6L^2c_2) &= \tfrac{1}{3}f_0L^3 + 2M_0L \\ EI(6L^2c_1 + 12L^3c_2) &= \tfrac{1}{4}f_0L^4 + 3M_0L^2 \end{aligned} \tag{2.90a}$$

or, in matrix form,

$$EI\begin{bmatrix} 4L & 6L^2 \\ 6L^2 & 12L^3 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{f_0L^3}{12}\begin{Bmatrix} 4 \\ 3L \end{Bmatrix} + M_0L\begin{Bmatrix} 2 \\ 3L \end{Bmatrix} \tag{2.90b}$$

Solving for c_1 and c_2, we obtain

$$c_1 = \frac{5f_0L^2 + 12M_0}{24EI}, \qquad c_2 = \frac{-f_0L}{12EI}$$

and the solution (2.88) becomes

$$w_2(x) = \frac{5f_0L^2 + 12M_0}{24EI}x^2 - \frac{f_0L}{12EI}x^3 \tag{2.91}$$

For the three-parameter approximation ($N = 3$), we obtain the matrix equation

$$EI\begin{bmatrix} 4 & 6L & 8L^2 \\ 6L & 12L^2 & 18L^3 \\ 8L^2 & 18L^3 & \frac{144}{5}L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{3}f_0L^2 + 2M_0 \\ \frac{1}{4}f_0L^3 + 3M_0L \\ \frac{1}{5}f_0L^4 + 4M_0L^2 \end{Bmatrix} \tag{2.92}$$

The solution of this when substituted into (2.88) for $N = 3$, gives

$$w_3(x) = \frac{f_0x^2}{24EI}(6L^2 - 4Lx + x^2) + \frac{M_0x^2}{2EI} \tag{2.93}$$

which coincides with the exact solution of (2.85) and (2.86). If we try to compute the four-parameter solution without knowing that the three-parameter solution is exact, the parameters c_j ($j > 3$) will be zero. Figure 2.4 shows a comparison of the Rayleigh–Ritz solution with the exact solution.

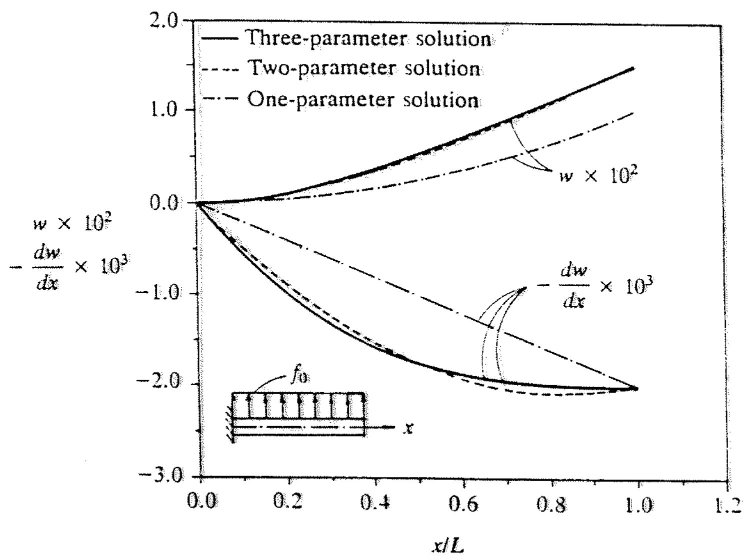
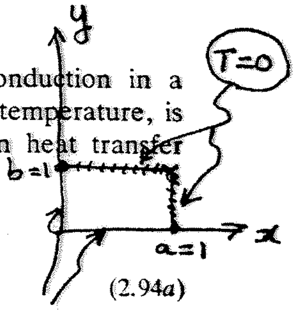


FIGURE 2.4
Comparison of the Rayleigh–Ritz solution with the exact solution of a cantilever beam under a uniform transverse load (Euler–Bernoulli beam theory).

The next example deals with two-dimensional heat conduction in a square region. Note that the dependent variable, namely the temperature, is denoted by T, consistent with the standard notation used in heat transfer books.



Example 2.6. Consider the Poisson equation in a unit square region:

$$-k\nabla^2 T = q_0 \quad \text{in } \Omega = \{(x, y): 0 < (x, y) < 1\} \tag{2.94a}$$

$$T = 0 \quad \text{on sides } x = 1 \text{ and } y = 1$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{on sides } x = 0 \text{ and } y = 0 \tag{2.94b}$$

$\frac{\partial T}{\partial n} = 0$

where q_0 is the rate of uniform heat generation in the region. The variational problem is of the form (see Example 2.3)

$$B(w, T) = l(w) \tag{2.95a}$$

where the bilinear and linear functionals are

$$B(w, T) = \int_0^1 \int_0^1 k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) dx\, dy + \beta \int_0^{b=1} w(x=1, y)\, T(x=1, y)\, dy$$

$$l(v) = \int_0^1 \int_0^1 w q_0 \, dx\, dy - \int_0^b w(0, y)\, \hat{q}\, dy + \beta \int_0^b w(a, y)\, T_\infty \, dy \tag{2.95b}$$

$\boxed{\text{Zero}}$

~~We consider an N-parameter approximation of the form~~

$$\cancel{T_N = \sum_{i,j=1}^{N} c_{ij} \cos \alpha_i x \cos \alpha_j y, \quad \alpha_i = \tfrac{1}{2}(2i - 1)\pi} \tag{2.96}$$

Note that (2.96) involves a double summation. Since the boundary conditions are homogeneous, we have $\phi_0 = 0$. Incidentally, ϕ_i also satisfies the natural boundary conditions of the problem. While the choice $\phi_i = \sin i\pi x \sin i\pi y$ meets the essential boundary conditions, it is not complete because it cannot be used to generate the solution that *does not* vanish on the sides $x = 0$ and $y = 0$. Hence, ϕ_i are not admissible.

The coefficients B_{ij} and F_i can be computed by substituting (2.96) into (2.95*b*). Since the double Fourier series has two summations [see (2.96)], we introduce the notation

$$B_{(ij)(kl)} = k \int_0^1 \int_0^1 [(\alpha_i \sin \alpha_i x \cos \alpha_j y)(\alpha_k \sin \alpha_k x \cos \alpha_l y) + (\alpha_j \cos \alpha_i x \sin \alpha_j y)(\alpha_l \cos \alpha_k x \sin \alpha_l y)]\, dx\, dy$$
$$= \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq l \\ \tfrac{1}{4}k(\alpha_i^2 + \alpha_j^2) & \text{if } i = k \text{ and } j = l \end{cases} \tag{2.97a}$$

$$F_{ij} = q_0 \int_0^1 \int_0^1 \cos \alpha_i x \cos \alpha_j y \, dx\, dy = \frac{q_0}{\alpha_i \alpha_j} \sin \alpha_i \sin \alpha_j \tag{2.97b}$$

In evaluating the integrals, the following orthogonality conditions were used

$$\int_0^1 \sin \alpha_i x \sin \alpha_j x \, dx = \begin{cases} 0 & \text{if } i \neq j \\ \tfrac{1}{2} & \text{if } i = j \end{cases}$$

$$\int_0^1 \cos \alpha_i x \cos \alpha_j x \, dx = \begin{cases} 0 & \text{if } i \neq j \\ \tfrac{1}{2} & \text{if } i = j \end{cases}$$

Owing to the diagonal form of the coefficient matrix (2.97*a*), we can readily solve for the coefficients c_{ij}:

$$c_{ij} = \frac{F_{ij}}{B_{(ij)(ij)}} = \frac{4q_0}{k} \frac{\sin \alpha_i \sin \alpha_j}{(\alpha_i^2 + \alpha_j^2)\alpha_i \alpha_j} \tag{2.98}$$

The one- and two-parameter Rayleigh–Ritz solutions are

$$T_1 = \frac{32 q_0}{k\pi^4} \cos \tfrac{1}{2}\pi x \cos \tfrac{1}{2}\pi y \tag{2.99}$$

$$T_2 = \frac{q_0}{k}[0.3285 \cos \tfrac{1}{2}\pi x \cos \tfrac{1}{2}\pi y - 0.0219(\cos \tfrac{1}{2}\pi x \cos \tfrac{3}{2}\pi y + \cos \tfrac{3}{2}\pi x \cos \tfrac{1}{2}\pi y) + 0.0041 \cos \tfrac{3}{2}\pi x \cos \tfrac{3}{2}\pi y] \tag{2.100}$$

If algebraic polynomials are to be used in the approximation of T, one can choose $\phi_1 = (1-x)(1-y)$ or $\phi_1 = (1-x^2)(1-y^2)$, both of which satisfy the (homogeneous) essential boundary conditions. However, the choice $\phi_1 = (1-x^2)(1-y^2)$ also meets the natural boundary conditions of the problem. The one-parameter Ritz solution for the choice $\phi_1 = (1-x^2)(1-y^2)$ is

$$T_1(x, y) = \frac{5q_0}{16k}(1-x^2)(1-y^2) \tag{2.101}$$

The exact solution of (2.94*a*, *b*) is

$$T(x, y) = \frac{q_0}{2k}\left[(1-y^2) + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos \alpha_n y \cosh \alpha_n x}{\alpha_n^3 \cosh \alpha_n}\right] \tag{2.102}$$

Basics: $\phi_i = \underbrace{(1-x^2)(1-y^2)}[a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + \dots]$

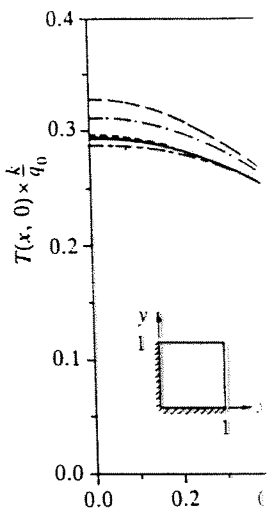


FIGURE 2.5
Comparison of the Rayle
(2.94) in two dimensions.

where $\alpha_n = \frac{1}{2}(2n-1)\pi$.
compared with the e
evaluated using 20 term

2.4.3 The Metho

As noted in Section
differential equatio
dependent variables
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possible to constru
variational form.
problems, includin
method, the weigh
approximation. Th
Rayleigh–Ritz met
independent set of
determine the para
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satisfies both the n
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but are required t