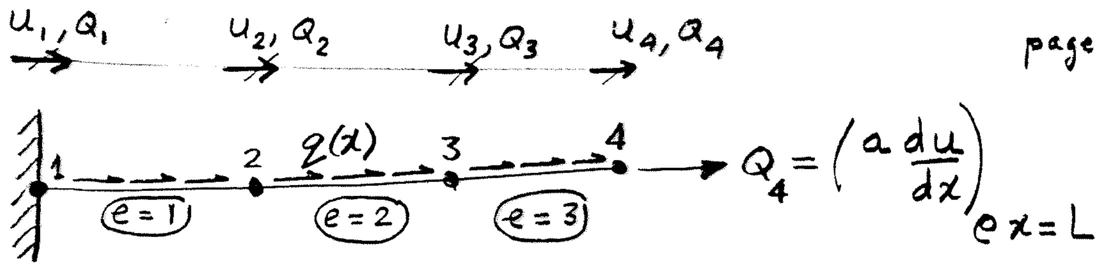
Figure 1



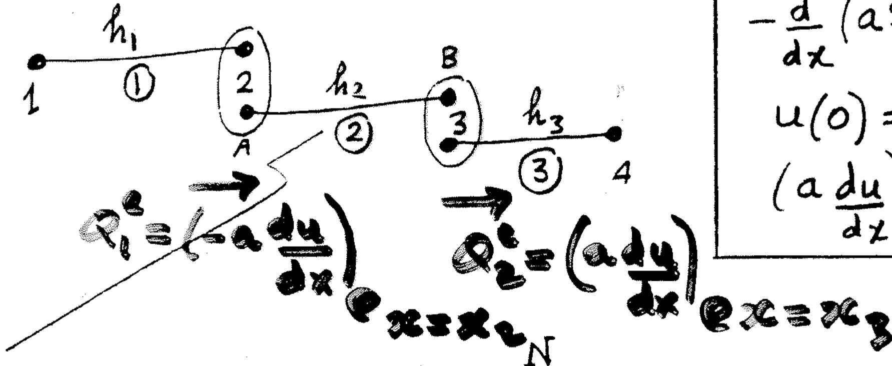
Given ODE

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

$$u(0) = u_0$$

$$\left(a\frac{du}{dx}\right)_{x=L} = Q_4$$

Figure 2



$$u^e = a + bx = [1, x]\begin{Bmatrix} a \\ b \end{Bmatrix} \equiv \sum_{i=1}^{N} u_i^e \, \phi_i^e(x) \qquad (1)$$

Applying 2 essential b.c. (**Non-homogeneous**):

$$\left.\begin{aligned} u_2 &= a + bx_2 \\ u_3 &= a + bx_3 \end{aligned}\right\} \Longrightarrow \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_2 \\ 1 & x_3 \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix} \Rightarrow \{u\} = [A]\vec{a}$$

Hence: $$\vec{a} \equiv \begin{Bmatrix} a \\ b \end{Bmatrix} = [A]^{-1}\{u\} \qquad (2)$$

Substitute Eq. (2) into Eq. (1):

$$u^e = \underbrace{[1, x] * [A]^{-1}}_{\text{shape, interpolation functions } \phi_i(x)}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \underbrace{[\phi_2(x), \phi_3(x)]}_{\text{shape functions}} * \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad (3)$$

where: $\phi_2(x) = \dfrac{x - x_3}{x_2 - x_3}$ and $\phi_3 = \dfrac{x - x_2}{x_3 - x_2}$ $\equiv$ Lagrange Interpolation Functions !

Properties:
- $\phi_i(x) = 1$ if $x = x_i$ & $\boxed{= 0, \text{elsewhere}}$
- $\sum \phi_i(x) = 1$
- $\sum \frac{d\phi_i(x)}{dx} = 0$

Questions [A] What happen to Lagrange shape functions if 3-node, or 4-node LINE ($\equiv$ 1-D) is used ??

[B] What happen when 4^{th} order ODE (beam) element is used

[C] ...

For a "typical finite element" #e :

Step 1 Set weighted Residual to zero

$$\int_{x_A = x_2}^{x_B = x_3} w\left[R \equiv -\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q\right] dx = 0 \qquad (5)$$

Step 2 Integrate by part ONCE (in this case!)

$$0 = \int_{x_A}^{x_B}\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - \Big[\underbrace{w}_{\substack{\text{primary}\\\text{variable}\\u}} * \underbrace{a\frac{du}{dx}}_{\substack{\text{secondary}\\\text{variable}\\Q}}\Big]_{x_A}^{x_B} \qquad (6)$$

Step 3 Imposing "actual, given" boundary conditions:

$$0 = \underbrace{\int_{x_A}^{x_B}\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx}_{B(w,u)} \boxed{- w(x_A)Q_A - w(x_B)Q_B} \qquad (7$$

$$B(w,u) = \ell(w) \qquad (8$$

Step 4 Let $w \equiv \phi_i^e(x)$ — see Eq (4) — (9

Let $u \equiv \sum_{j=1}^{N} u_j^e\, \phi_j^e(x)$ — see Eq. (1) — (10

Then, Eq. (7) becomes:

$$0 = \int_{x_A}^{x_B}\left[a\frac{d\phi_i^e}{dx} * \frac{d}{dx}\left(\sum_{j=1}^{N} u_j^e \phi_j^e\right) + c\,\phi_i^e\left(\sum_{j=1}^{N} u_j^e \phi_j^e\right) - \phi_i^e q\right]dx \;\; \boxed{- \sum_{j=1}^{N} \phi_i^e(ex_j) * Q_j} \qquad (1$$

$$Q_i \equiv$$

using properties of shape functions

Mor general case (if Line element has more than 2-nodes) or, Eq. (11) can be symbolically represented as:

$$0 = \left(\sum_{j=1}^{N} K_{ij}^{e} * u_{j}^{e} \right) - f_{i}^{e} - Q_{i}^{e} \quad \text{———} \quad (12)$$

where $i = 1, 2, \ldots, N$

where:

$$[K_{ij}^{e}] \equiv \int_{x_A}^{x_B} \left(a \frac{d\phi_i^e}{dx} * \frac{d\phi_j^e}{dx} + c\, \phi_i^e \phi_j^e \right) dx = \text{"stiffness matrix"} \quad \text{———} \quad (13)$$

$$\equiv B(\phi_i^e, \phi_j^e)$$

$$\{f_i^e\} \equiv \int_{x_A}^{x_B} q\, \phi_i^e \, dx \equiv \ell(\phi_i^e) \equiv \text{"Equivalent Nodal Load vector"} \quad \text{———} \quad (14)$$

$$\{Q_i^e\} \equiv \quad \text{nodal load vector} \quad \text{———} \quad (15)$$

Eq. (12) can be further abbreviated as:

$$[k^e]\{u^e\} = \{f^e\} + \{Q^e\} \quad \text{———} \quad (16)$$

From Figure 2, for each typical e^{th} finite element, assuming 2-term approx. is used (say $N=2$), then (assuming "a & q" given in the ODE are <u>constants</u> within an e^{th} finite element):

$$\left(\frac{a^e}{h^e}\right) * \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2^e \\ u_3^e \end{Bmatrix} = \left(\frac{q^e h^e}{2}\right) * \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \quad \text{——— (17)}$$

Remarks

#(1) Refering to Figure 2, a typical e^{th} F.E. will give

- 2 Equations (see Eq. 12, with $i, j = 1, 2, \ldots, N=2$)
- 4 Unknowns (for a typical $(e=2)^{th}$ F.E.) which are $u_2^{(2)}$, $u_3^{(2)}$, $Q_2^{(2)}$ & $Q_3^{(2)}$ = force

#(2) Thus, for "entire" domain (see Figure 1), which has 3 F.E., one has:

- (2 Equations per F.E.) * (3 F.E.) = $\boxed{\text{6 Eqs.}}$
- $\boxed{\text{12 unknowns}}$ $\left(= u_1^{(1)}, u_2^{(1)}, Q_1^{(1)}, Q_2^{(1)}, \ldots, u_3^{(3)}, u_4^{(3)}, Q_3^{(3)}, Q_4^{(3)}\right)$

#(3) Additional Eqs. can be obtained from "system" b.c., compatibility @ "common" nodes & applied "nodal loads"

$$\left.\begin{array}{l} U_1 = 0 \; ; \quad u_2^{(1)} = u_2^{(2)}, \; u_3^{(2)} = u_3^{(3)} \; ; \\ Q_2^{(1)} + Q_2^{(2)} = 0, \; Q_3^{(2)} + Q_3^{(3)} = 0 . \; Q^{(3)} = Q_0 \end{array}\right\} \text{during ASSEMBL... process}$$

#4 Assembly Process:

Assuming $a^e = h^e = q^e = 1$ (see Eq. 17)

Assuming $Q_4 = 2.8$

Then; Eq. (16) becomes:

$$\underset{4\times4}{[K]}\ \underset{4\times1}{\{U\}} = \underset{4\times1}{\{F\}} + \underset{4\times1}{\{Q\}} \qquad (18)$$

where $\underset{4\times4}{[K]} \equiv$ "system" stiffness matrix $= \sum_{e=1}^{3\text{ elements}} \left[k^e\right]_{2\times2}$ — (

$$\underset{4\times1}{\{F\}} \equiv \sum_{e=1}^{3} \{F^e\}_{2\times1} \qquad (20)$$

$$\underset{4\times1}{\{Q\}} \equiv \sum_{e=1}^{3} \{Q^e\}_{2\times1} \qquad (21)$$

In this particular example, one has:

singular matrix ☹

$$\underset{4\times4}{[K]} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & -1 & & \\ 2 & -1 & 1 \ \boxed{1} & \boxed{-1} & \\ 3 & & \boxed{-1} & \triangle 1 \ \boxed{1} & \triangle -1 \\ 4 & & & \triangle -1 & \triangle 1 \end{array} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad (2$$

Notes:

○ ≡ element 1

□ ≡ element 2

△ ≡ element 3

$$\{F\} = \frac{1}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}0\\0\\1\\1\end{Bmatrix} = \frac{1}{2}\begin{Bmatrix}1\\2\\2\\1\end{Bmatrix} \qquad (23)$$

$$\{Q\} = \begin{Bmatrix}Q_1\\Q_2\\0\\0\end{Bmatrix} + \begin{Bmatrix}0\\Q_2\\Q_3\\0\end{Bmatrix} + \begin{Bmatrix}0\\0\\Q_3\\Q_4\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\2.8\end{Bmatrix} \qquad (24)$$

After imposing "system b.c.", $\left.\begin{matrix}\text{Eq. (18)} \\ \text{\& Eqs. (22→24)}\end{matrix}\right\}$ becomes:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 = k_1 = 0 \\ U_2 \ ?? \\ U_3 \ ?? \\ U_4 \ ?? \end{Bmatrix} = \begin{Bmatrix} Q_1 \ ?? \\ \frac{1}{2}2 + 0 \\ \frac{1}{2}2 + 0 \\ \frac{1}{2}1 + 2.8 \end{Bmatrix} \qquad (25)$$

Eq. (25) can be transformed to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} k_1 = 0 \\ \frac{1}{2}2 \\ \frac{1}{2}2 \\ 3.3 \end{Bmatrix} \qquad (26)$$

??

- System of sparse, symmetrical, positive definite linear Eq. (26) can be solved by skyline (or sparse) Choleski factorization + forward/backward solution.
- Having solved for $U_1 \rightarrow U_4$, the <u>unknown</u> "reaction force" Q_1 can be solved by refering to the <u>first</u> Equation of Eq. (25).