$$\int_{x_A}^{x_B} u\,dv = [uv]_{x_A}^{x_B} - \int v\,du$$

Given the ODE:

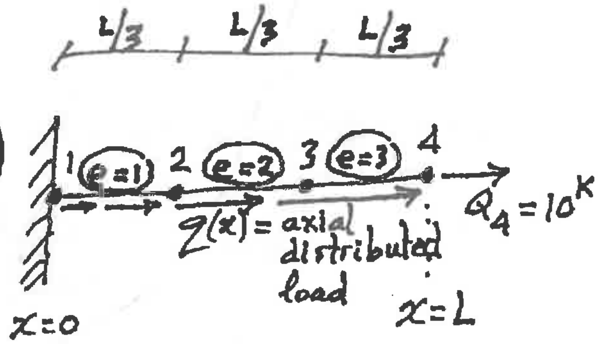
$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0, \quad \text{where } \Omega : x = 0 \to L$$

"Dirichlet" Geometric b.c. with bc

$$\longrightarrow u(@\,x=0) = u_0 \ (\text{say} = 0)$$

"Natural" b.c. Neuman

$$\longrightarrow \left(a\frac{du}{dx}\right)_{@\,x=L} = Q_4 \ (\text{say} = 10^K)$$



<u>Weak formulation</u>

$\boxed{\text{step 1}}$

$$0 = \int_{x=0}^{L} w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q\right]dx \ \text{-----}\!\!\to \text{Eq.(1)}$$

$\boxed{\text{step 2}}$ Integrate 1st term by part (ONCE)

$$0 = \int\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - \left[w * a\frac{du}{dx}\right]_{x_A=0}^{x_B=L} \ \text{---}\!\!\to \text{Eq.(2)}$$

primary variable u

Secondary variable Q

$\boxed{\text{Step 3}}$ Applying the given "natural b.c."

$$0 = \int\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - \left\{\left[w * a\frac{du}{dx}\right]^{x_B} + \left[w * \left(a\frac{du}{dx}\right)\right]_{x_A}\right\}$$

$$0 = \int\left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wq\right)dx - \left\{w(@x_A) * (Q_A) + w(@x_B) * (Q_B)\right\}$$

$$0 = \int\left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right)dx - \left\{\int wq\,dx + w(@x_A)\,Q_A + w(@x_B)\,Q_B\right\}$$

$$0 = B(w,u) - l(w) \ \text{---}\!\!\to \text{Eq.(3)}$$

Thus,

$$B(w,u) = \int\left(a\,\phi_i' * \phi_j' + c\,\phi_i * \phi_j\right)dx * \vec{c}_j \qquad [K_{ij}]$$

$$l(w) = \int \phi_i q\,dx + \phi_i(@x_A)\,Q_A + \phi_i(@x_B)\,Q_B \equiv \vec{F}_i^e$$

$$[K_{ij}^e]\,\vec{c}_j = \vec{F}_i^e$$

$\boxed{\text{step 4}}$ Garlekin Finite Element ~~(FE)~~ Approx. Method

$N = \#$ dof per FE $= 2$ for this specific example (based on the ODE $2m = 2$)

$$u^{(e)}(x) \approx \sum_{j=1}^{N} u_j^{(e)} \phi_j^{(e)}(x) = u_1^{(e)}\phi_1^{(e)} + u_2^{(e)}\phi_2^{(e)} + \cdots \text{-----} \rightarrow \text{Eq.}(4)$$

where $\phi_j^{(e)}(x)$ = selected function to satisfy "Geometric b.c"

over a "typical e^{th} Finite Element" say $e^{th} = 2$

For this specific example,

$$u^{(e)}(x) = a + bx \cancel{+ cx^2 + \cdots} = \underset{1\times 2}{[1, x]} \underset{2\times 1}{\begin{Bmatrix} a \\ b \end{Bmatrix}} \equiv \sum_{j=1}^{N=2} u_j^{(e)} \phi_j^{(e)}(x) \text{----} \rightarrow \text{Eq.}(11)$$

Applying 2 "Geometric b.c" for FE $e=2$ to get:

@ mode 2 : $x = x_2$ and $u^{(e)} = u_2 \Rightarrow \boxed{u_2 = a + b x_2}$

@ mode 3 : $x = x_3$ and $u^{(e)} = u_3 \Rightarrow \boxed{u_3 = a + b x_3}$ $\text{---} \rightarrow \text{Eq.}(12)$

or $$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} \text{-----------} \rightarrow \text{Eq.}(13)$$

or $$\vec{u} = [A] * \vec{a} \text{-----------} \rightarrow \text{Eq.}(14)$$

Hence: $$\vec{a} \equiv \begin{Bmatrix} a \\ b \end{Bmatrix} = [A]^{-1} * \vec{u} \text{--------} \rightarrow \text{Eq.}(15)$$

Substitute Eq. (15) into Eq. (11) to get:

$$u^{(e)}(x) = [1, x][A]^{-1} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \equiv \sum u_j^{(e)} \phi_j^{(e)}(x)$$

$$u^{(e)}(x) = [\phi_2(x), \phi_3(x)] * \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

FE Shape functions

$$\phi_2(x) = \frac{x - x_3}{x_2 - x_3}$$

$$\phi_3(x) = \frac{x - x_2}{x_3 - x_2}$$

properties

$$\phi_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases}$$

$$\sum \phi_i(x) = 1$$

$$\sum_i \phi_i'(x) = 0$$

etc....

Assemble elements' contributions to form system Equations

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \boxed{K^{(1)}_{11}} & K^{(1)}_{12} & \cdot & \cdot \\ K^{(1)}_{21} & \boxed{K^{(1)}_{22}+K^{(2)}_{22}} & K^{(2)}_{23} & \cdot \\ & K^{(2)}_{32} & \boxed{K^{(2)}_{33}+K^{(3)}_{33}} & K^{(3)}_{34} \\ \text{SYM} & & K^{(3)}_{43} & \boxed{K^{(3)}_{44}} \end{bmatrix} \end{array} \begin{Bmatrix} Z_1 = 0'' \\ Z_2 \; ?? \\ Z_3 \; ?? \\ Z_4 \; ?? \end{Bmatrix} \neq k_1 \quad = \begin{Bmatrix} R_1 \; ?? \\ Q_2 = k_2 \\ Q_3 = k_3 \\ Q_4 = k_4 \end{Bmatrix}$$

$$[K]\{Z\} = \{\vec{F}\}$$

Singular Matrix

Mixing between unknown Reaction & known forces

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K^{(1)}_{22}+K^{(2)}_{22} & K^{(2)}_{23} & \cdot \\ 0 & K^{(2)}_{32} & K^{(2)}_{33}+K^{(3)}_{33} & K^{(3)}_{34} \\ 0 & \cdot & K^{(3)}_{43} & K^{(3)}_{44} \end{bmatrix} \end{array} \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \begin{Bmatrix} k_1 = 0'' \\ k_2 - K^{(1)}_{21} k_1 \\ k_3 - \cancel{K_{31}} k_1 \\ k_4 - \cancel{K_{41}} k_1 \end{Bmatrix}$$

zeroes

All known quantities

Non-singular

$$[K^*]\{Z\} = \{F^*\}$$

Then, apply Cholesky Method !!

$\boxed{\text{step 4}}$.

Ritz Numerical Approx. Method

$$u(x) \approx \sum_{j=1}^{N = \text{say } 2, \text{ or } 3 \cdots} c_j \, \phi_j(x) = c_1 \phi_1(x) + c_2 \phi_2(x) \quad \text{------} \rightarrow \text{Eq. (4)}$$

where $\phi_j(x)$ = selected function to satisfy "Geometric b.c" over the "entire domain" ($x = 0 \rightarrow L$).

For this example,

$$\left.\begin{aligned} \phi_1(x) &= (x - 0) = x \\ \text{So: } \phi_2(x) &= x\,\phi_1(x) = x^2 \\ \cancel{\phi_3(x)} &\cancel{= x^2 \phi_1(x) = x^3, \text{ etc...}} \end{aligned}\right\} \text{-----} \rightarrow \text{Eq. (5)}$$

From Eq. (3):

$$B(w,u) \equiv \int_{x_A=0}^{x_B=L} \left[a \frac{d\phi_i}{dx} \frac{d}{dx}\left(\cancel{c_j}\,\phi_j\right) + c\,\phi_i \left(\cancel{c_j}\,\phi_j\right) \right] dx$$

$$B(w,u) = K_{ij}\, c_j = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

where:

$$\underbrace{K_{ij} \equiv \int_{x_A=0}^{x_B=L} \left[a \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + c\,\phi_i \phi_j \right] dx} \quad \text{with } i,j = 1 \,\&\, 2$$

For example

$$K_{12} \equiv \int_{x_A}^{x_B} \left[a \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} + c\,\phi_1 \phi_2 \right] dx$$

Also:

$$l(w) \equiv \int_{x_A=0}^{x_B=L} \phi_i\, q \, dx + \cancel{w(@x_A)\, Q_A} + \cancel{w(@x_B)}\, Q_B = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$Q_B \rightarrow Q_A = 10^K$

$\phi_i(@x_B)$

Given distributed axial load function

~~Zero, see Given "Geometric b.c."~~

Thus:

$$\overbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}}^{B(w,u)} - \overbrace{\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}}^{l(w)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \boxed{\text{solve for } c_1 \,\&\, c_2}$$

How to Handle Non-Homogeneous Geometrical (Dirichlet) Boundary Conditions ??

Given:

@ $x = x_0$; $u(@ x_0) = u_0$ Known $\neq 0$

Step 4 (using Ritz Method)

$$u(x) = \left(\sum_{j=1}^{N = \#\text{ terms used}} c_j \underbrace{\phi_j(x)} \right) + \boxed{\phi_0(x)}$$

Hence:

$$u(@ x_0) = \text{Zero} + u_0$$

satisfying "Homogeneous Portion" of Geometrical b.c.

see J.N. Reddy's book, page 42

can be selected such that:

Ⓐ satisfying "Non-Homogeneous Portion" of Geometric b.c.

@ $x = x_0 = L$

$\phi_0 = 0.3''$

Ⓑ ϕ_j are required to satisfy HOMO. form of same boundary conditions

At $x = 0$, $u = 0$? Yes

Example: @ $x = 0 \Rightarrow u(@0) = 0$

@ $x = L = x_0 \Rightarrow u(@L) = 0.3'' = u_0$

Hence, try
$$\phi_j(x) = (x - 0)(x - L)$$
$$\phi_0(x) \underset{\text{say}}{=} x$$

still need this term for Homo. b.c.

Thus:

$$u(x) = \left[\sum_{j=1}^{N\text{-terms}} c_j \underbrace{(x-0)(x-L)}_{\phi_j} \right] + \underbrace{\left(\frac{0.3x}{L} \right)}_{\phi_0}$$

At $x = x_0 = L$; $u = u_0 = 0.3$? YES

Step 4 (using Finite Element Galerkin Method)

No-thing changes ! ☺

Example(s) about Bilinear, Symmetrical operator B(w,u), and Linear operator l(w)

The operator **<u>B(w, u) is said to be BILINEAR if</u>** it is LINEAR in each of its arguments w, and u, such as:

B(alpha * w1 + beta * w2, u) = alpha B(w1,u) + beta B(w2,u); hence LINEAR in the FIRST argument (1)

B(w, alpha * u1 + beta * u2) = alpha B(w, u1) + beta B(w, u2) .... (2)

The operator B(w, u) is said to be SYMMETRICAL if B(w, u) = B(u, w) .... (3)

The operator **<u>l(w) is said to be LINEAR in w if and only if</u>** it satisfies the following relation:

l(alpha * w + beta * u) = alpha * l(w) + beta * l(u) .... (4)

<u>J.N. Reddy's 3-rd Edition textbook, page #65</u>

" .... Whenever B(w,u) is BILINEAR & SYMMETRIC, and l(w) is LINEAR, then the functional I(u) associated with B(w, u) = l(w), can be given as I(u) = 0.5 * B(u, u) – l(u) .... " ...... (2.4.19)

<u>An Example</u>: [J.N. Reddy's 3rd Edition, pp. 64]

Given the following 2 operators

B(w, u) = Integraf$_0^L$ (a * w' * u') dx and $l(w) = \int_0^L \{w\}\, dx + w(@L)\, Q_L$ .... (5)

<u>If we "swap" w = u, and u = w, then:</u>

<u>B(u, w) = integral (a * u' * w') dx = B(w, u). Thus, operator B(w,u) is SYMMETRICAL !!</u>

B($\underbrace{\text{alpha * w1 + beta * w2}}_{w}$, u) = Integral$_0^L$ (a * {alpha * w1 + beta * w2}' * u ') dx $\equiv \{w\}'$

= Integral$_0^L$ (a * {alpha * w1' + beta * w2'} * u ') dx

= Integral$_0^L$ (a * {alpha * w1' } * u ') dx + Integral$_0^L$ (a * {beta * w2'} * u ') dx

= alpha * Integral$_0^L$ (a * {w1' } * u ') dx + beta * Integral$_0^L$ (a * {w2'} * u ') dx

B(alpha * w1 + beta * w2, u) = alpha * B(w1, u) + beta * B(w2, u).

Hence, B(w,u) is LINEAR in the FIRST argument.

Similarly, we can prove that B(w,u) is LINEAR in the SECOND argument.

Notes: STEP 4 = Minimizing the functional $I(u)$

since operator $B(w,u) = \begin{cases} \text{SYM.} \iff B(w,u) = B(u,w) \\ \text{Bilinear} \iff B(\alpha w_1 + \beta w_2, u) = \alpha B(w_1,u) + \beta B(w_2,u) \end{cases}$

& $l(w)$ = linear operator $\iff l(\alpha w + \beta v) = \alpha l(w) + \beta l(v)$

Hence

$$I(u) \equiv \pi(u) = \frac{1}{2} B(u,u) - l(u)$$

$$= \frac{1}{2}\int\left[a\left(\frac{du}{dx}\right)^2 + c u^2\right]dx - \left\{\int u q\, dx + \underbrace{\cancel{u(@x_A)}\,Q_A}_{\text{zero, since } u(@x_A) \text{ is prescribed}} + u(@x_B)\underbrace{Q_B}_{= Q_4 = 10^k}\right\}$$

$$I(u) = \frac{1}{2}\left\{\int_0^L \left[a\left(\frac{du}{dx}\right)^2 + c u^2 - 2uq\right]dx\right\} - \underbrace{\cancel{u(@x_B)}}_{u(@L)} * 10^k$$

Now, $u(x) \approx c_1 \phi_1(x) + c_2 \phi_2(x)$

so: $\dfrac{du}{dx} \approx c_1 \phi_1'(x) + c_2 \phi_2'(x)$

$$I(u) = \frac{1}{2}\left\{\int_0^L \left(a\left[c_1\phi_1' + c_2\phi_2'\right]^2 + c\left[c_1\phi_1 + c_2\phi_2\right]^2 - 2\left[c_1\phi_1 + c_2\phi_2\right]q\right)dx\right\}$$

$$- 10^k\left[c_1\phi_1(@L) + c_2\phi_2(@L)\right]$$

Set:

$$\frac{\partial I}{\partial c_1} = 0$$

$$\frac{\partial I}{\partial c_2} = 0$$

Hence:

$$\frac{\partial I}{\partial c_1} = 0 = \cancel{\frac{1}{2}}\left\{ \int_0^L \left(\cancel{2} a\left[c_1\phi_1' + c_2\phi_2'\right]\phi_1' + \cancel{2} c\left[c_1\phi_1 + c_2\phi_2\right]\phi_1 - \cancel{2}\left[\phi_1\right] q \right) dx \right\} - 10^K\left[\phi_1(@L)\right]$$

$$\frac{\partial I}{\partial c_2} = 0 = \cancel{\frac{1}{2}}\left\{ \int_0^L \left(\cancel{2} a\left[c_1\phi_1' + c_2\phi_2'\right]\phi_2' + \cancel{2} c\left[c_1\phi_1 + c_2\phi_2\right]\phi_2 - \cancel{2}\left[\phi_2\right] q \right) dx \right\} - 10^K\left[\phi_2(@L)\right]$$

The above 2 Eqs. can be re-written as:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \int_0^L (a\phi_1'\phi_1' + c\phi_1\phi_1)\,dx & \int_0^L (a\phi_1'\phi_2' + c\phi_1\phi_2)\,dx \\ \int_0^L (a\phi_2'\phi_1' + c\phi_2\phi_1)\,dx & \int_0^L (a\phi_2'\phi_2' + c\phi_2\phi_2)\,dx \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} - \begin{Bmatrix} \int_0^L \phi_1 q\,dx + 10^K\phi_1(@L) \\ \int_0^L \phi_2 q\,dx + 10^K\phi_2(@L) \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} - \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$\boxed{\text{Same answers/formulas as Ritz Method ☺, see page 4!}}$