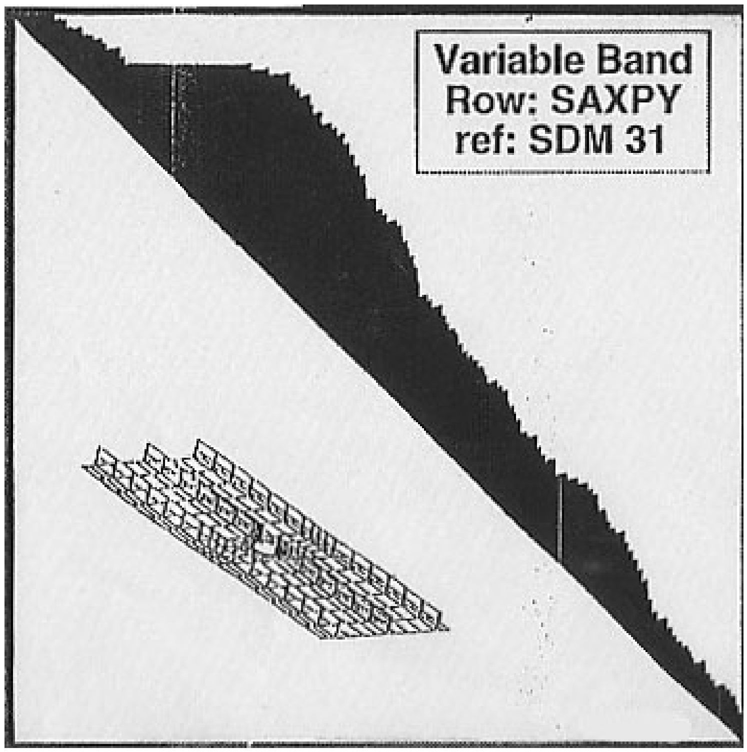
$$
\left[\, s \,\right] = \begin{bmatrix}
S_{11} & S_{12} & 0 & S_{14} & 0 & 0 & 0 & 0 & 0 \\
 & S_{22} & S_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & S_{33} & S_{34} & 0 & S_{36} & 0 & 0 & 0 \\
 & & & S_{44} & S_{45} & S_{46} & 0 & 0 & 0 \\
 & & & & S_{55} & S_{56} & 0 & S_{58} & 0 \\
 & \text{sym.} & & & & S_{66} & S_{67} & 0 & S_{69} \\
 & & & & & & S_{77} & S_{78} & 0 \\
 & & & & & & & S_{88} & S_{89} \\
 & & & & & & & & S_{99}
\end{bmatrix}
$$

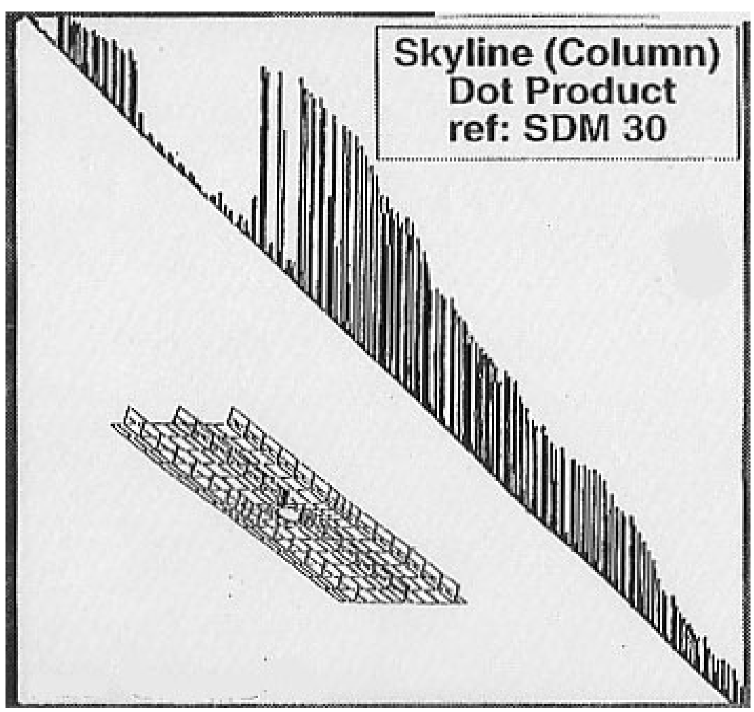
<u>STORAGE SCHEMES :</u>

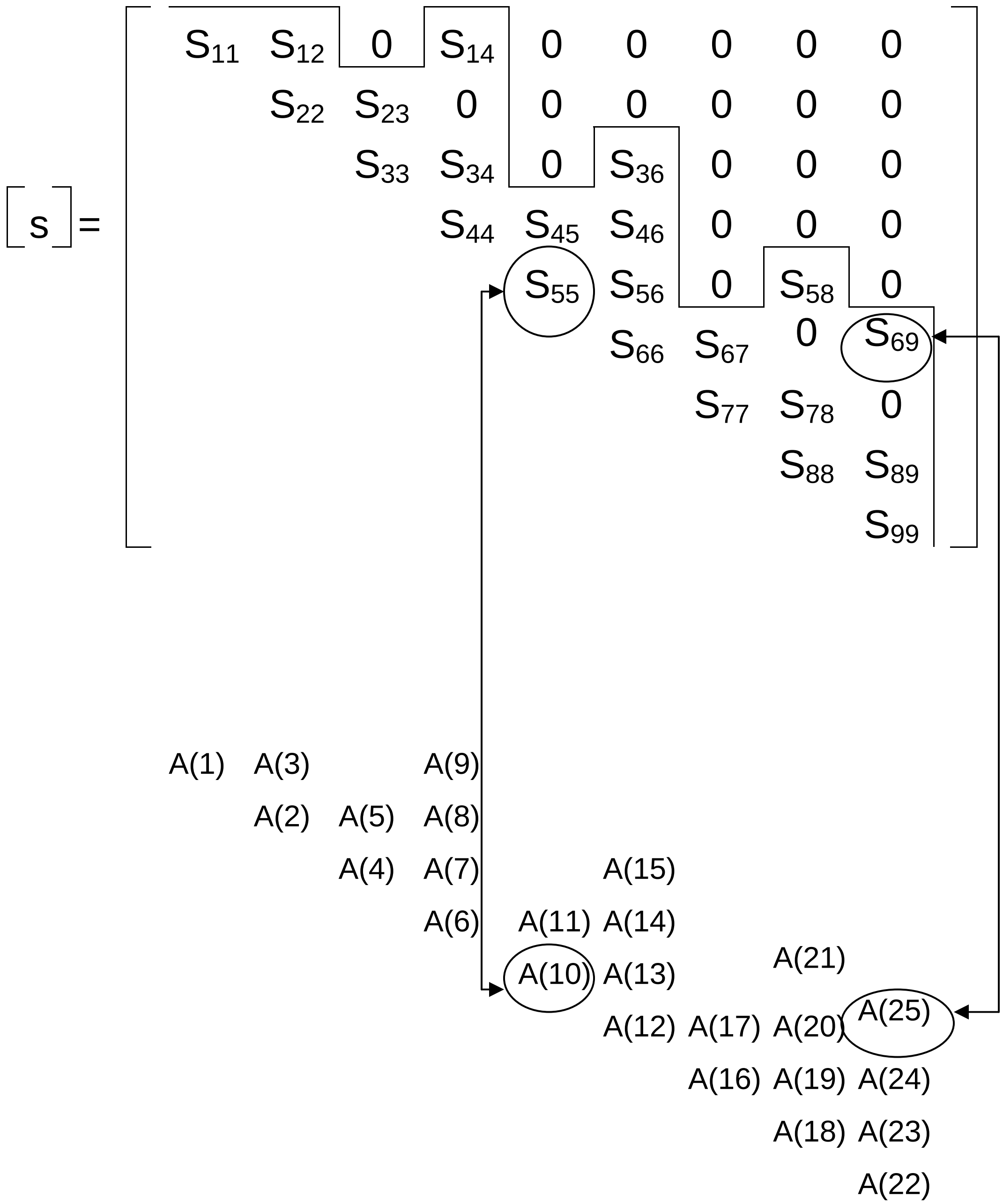
• FULL	81 WORDS
• SYMMETRIC	45 WORDS
• BANDED	36 WORDS
• VARIABLE BAND	28 WORDS
• SKYLINE	25 WORDS

SPARSE

Matrix Storage Methods
2910 Equation Stiffness Panel







SKYTLINE STORAGE SCHEME FOR STIFFNESS MATRIX

$$
\text{COLH}\quad \begin{bmatrix} 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{bmatrix} = \begin{Bmatrix} 0\\1\\1\\3\\1\\3\\1\\3\\3 \end{Bmatrix} = \text{Column Height = "Known"}
$$

$$
\text{MAXA}\quad \begin{bmatrix} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10=N+1 \end{bmatrix} = \begin{Bmatrix} 1\\2\\4\\6\\10\\12\\16\\18\\22\\26 \end{Bmatrix} = \text{Location of Diagonal Terms}
$$

MAXA (1) = 1 ALWAYS !

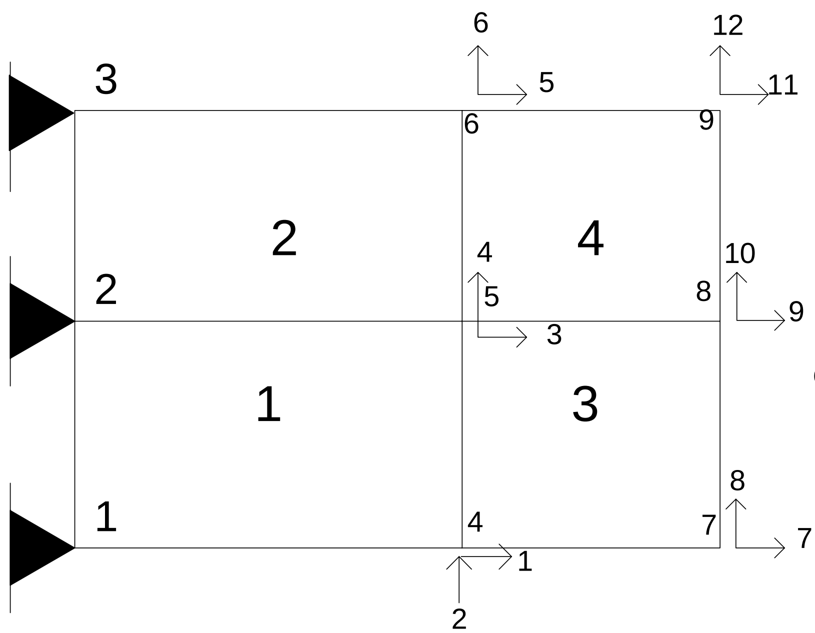
MAXA (I+1) = MAXA(I) + COLH(I) + 1

THUS: S_{IJ} = A[MAXA(J) + J – I]

Example: S_{55}= A[MAXA(5) + 5 - 5] = A [10]

S_{69}= A[MAXA(9) + 9 - 6] = A [25]

How To Find The Column Height of Each Column
Example:



NEL= Number of Elements = 4

NNODES = Number of Nodes = 9

$$
\left[\ \mathrm{ID}\ \right] = \begin{array}{c} \\ T_x \\ T_y \\ T_z \\ R_x \\ R_y \\ R_z \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

Convention :

1 If the Degree of Freedom (DOF) is fixed

0 If the DOF is free to move

$$
[ID] = \begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 3 & 5 & 7 & 9 & 11 \\
0 & 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} \\ Tx \\ T_y \\ Tz \\ R_x \\ R_y \\ R_z \end{array}
$$

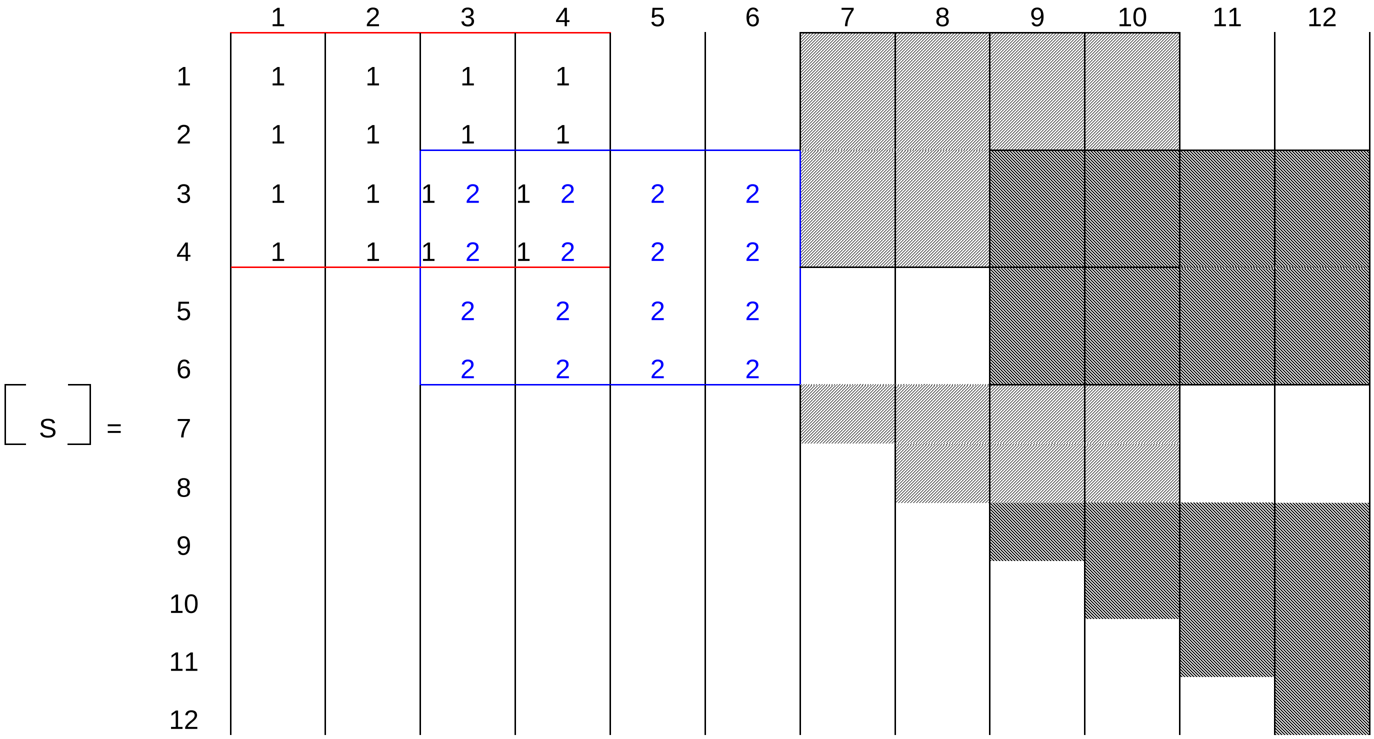
The following are element connectivities:

$LM^{(1)}$= nodes 5,2,1,4 = (3,4,0,0,0,0,1,2)

$LM^{(2)}$= nodes 6,3,2,5 = (5,6,0,0,0,0,3,4)

$LM^{(3)}$= nodes 8,5,4,7 = (9,10,3,4,1,2,7,8)

$LM^{(4)}$= nodes 9,6,5,8 = (11,12,5,6,3,4,9,10)



Thus, the column height for each column is given as:

$$\text{COLH} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 2 \\ 3 \\ 6 \\ 7 \\ 8 \\ 9 \\ 8 \\ 9 \end{Bmatrix}$$

The following is a skeletal Fortran statements to compute the column height of each column:

DO 1 I = 1, NEL

C… For I = 1, the smallest DOF = 1, hence

$$\text{COLH}\begin{Bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 3-1 \\ 4-1 \\ 1-1 \\ 2-1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{Bmatrix}$$

C…For element I= 2, the smallest DOF = 3, hence

$$\text{COLH}\begin{Bmatrix} 5 \\ 6 \\ 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 5-3 \\ 6-3 \\ 3-3 \\ 4-3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ \cancel{0} \rightarrow 2 \\ \cancel{1} \rightarrow 3 \end{Bmatrix}$$

C…For element I= 3, the smallest DOF = 1, hence

$$\text{COLH}\begin{Bmatrix} 9 \\ 10 \\ 3 \\ 4 \\ 1 \\ 2 \\ 7 \\ 8 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 9 \\ 2 \\ 3 \\ 0 \\ 1 \\ 6 \\ 7 \end{Bmatrix}$$

C…For element I= 4, the smallest DOF = 3, hence

$$\text{COLH}\begin{Bmatrix} 11 \\ 12 \\ 5 \\ 6 \\ 3 \\ 4 \\ 9 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 9 \\ 2 \\ 3 \\ \cancel{0} \rightarrow 2 \\ \cancel{1} \rightarrow 3 \\ \cancel{6} \rightarrow 8 \\ \cancel{7} \rightarrow 9 \end{Bmatrix}$$

1 continue

SEQUENTIAL CHOLESKI METHOD

$$[S]\{\overset{?}{Z}\} = \{F\}$$

Step 1. Factorization

$$[S] = [L][U] = [U]^T[U]$$

Hence

$$[U]^T\left([U]\{Z\}\right) = \{F\}$$

Step 2. Forward Substitution, Solve For $\{y\}$

$$[U]^T\left[\{Y\}\right] = \{F\}$$

Step3. Backward Substitution, Solve For $\{Z\}$

$$[U]\{Z\} = \{Y\}$$

TO FIND FACTORIZED MATRIX [U]

Example known

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Thus:

s_{11}	=	U_{11}^2	U_{11}	=	SQRT (S_{11})	1st
s_{12}	=	$U_{11}\,U_{12}$	U_{12}	=	$s_{12}/\,U_{11}$	2nd
s_{13}	=	$U_{11}\,U_{13}$	U_{13}	=	$s_{13}/\,U_{11}$	4th
s_{22}	=	$U_{12}^2 + U_{22}^2$	U_{22}	=	SQRT$(S_{22}-U_{12}^2)$	3rd
s_{23}	=	$U_{12}\,U_{13}+U_{22}\,U_{23}$	U_{23}	=	$(S_{23}-U_{12}U_{13})/U_{22}$	5th
s_{33}	=	$U_{13}^2 + U_{23}^2 + U_{33}^2$	U_{33}	=	SQRT$(S_{33}-U_{13}^2-U_{23}^2)$	6th

Notes:

[U] overwrite [S] !

Column Oriented Approach

PV- Solve (INCORE Version)

In the sequential Choleski method, a symmetric, Positive definite stiffness matrix, [K] , can be decomposed as

$$[K] = [U]^T[U] \tag{1}$$

with the coefficients of the upper-triangular matrix, [U]:

$$U_{ij} = 0 \text{ for } i > j \tag{2}$$

$$u_{11} = \sqrt{K_{11}} \quad u_{1j} = \frac{K_{1j}}{u_{11}} \quad \text{for} \quad j \geq 1 \tag{3}$$

$$u_{ii} = \sqrt{K_{ii} - \sum_{k=1}^{i-1} (U_{ki})^2} \quad \text{for} \quad i \geq 1 \tag{4}$$

$$u_{ij} = \frac{K_{ij} - \sum_{k=1}^{i-1} U_{ki} U_{kj}}{U_{ii}} \quad \text{for} \quad i, j \geq 1 \tag{5}$$

$$u_{57} = \frac{k_{57} - u_{15} \cdot u_{17} - u_{25} \cdot u_{27} - u_{35} \cdot u_{37} - u_{45} \cdot u_{47}}{u_{55}} \tag{8}$$

$$
[S] = \begin{array}{cc}
\begin{matrix} \phantom{S_{11}} & \phantom{S_{12}} & \phantom{S_{13}} & I=4 & \phantom{S_{15}} & \phantom{S_{16}} & J=7 & \phantom{S_{18}} & \phantom{S_{19}} \\ & & & \downarrow & & & \downarrow & & \end{matrix} & \\
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\
 & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\
 & & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\
 & & & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\
 & & & & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\
\text{sym} & & & & & S_{66} & S_{67} & S_{68} & S_{69} \\
 & & & & & & S_{77} & S_{78} & S_{79} \\
 & & & & & & & S_{88} & S_{89} \\
 & & & & & & & & S_{99}
\end{bmatrix} &
\begin{matrix} \\ \\ \leftarrow \text{Row } I-1=3 \\ \leftarrow \text{Row } I=4 \\ \\ \\ \leftarrow \text{Row } J=7 \\ \\ \\ \end{matrix}
\end{array}
$$

$$
U_{47} = \frac{S_{47} - \sum_{K=1}^{I-1} \left(U_{14}U_{17} + U_{24}U_{27} + U_{34}U_{37} \right)}{U_{44}}
$$

U_{77} =?

U_{11} = SQRT (S_{11}) col #

Do 1 J = 2,n (say J = 7th Column)

Do 2 I = Top Row of Col J , Row J (say I = 4th Row)

Do 3 k = Top Row of Col I , Row I –1

3 Compute $\sum U_{KI} U_{KJ}$

Compute $S_{IJ} - \sum U_{KI} U_{KJ}$

.

.

2 continue

1 continue

Version 1 Basic (Column oriented) Choleski

U_{11} = SQRT (S_{11})

DO 1 J = 2, N (say J=7th Column)

DO 2 I = top row of Col J, Row J (say I = 4th Row)

Sum 1 = 0

DO 3 K = Top Row of Col I ,Row I-1

3 Sum 1 = Sum 1 + U_{KI} * U_{KJ}

U_{IJ} = S_{IJ} – SUM1

$$\boxed{\begin{array}{l} \text{If (I.EQ.J) Then} \\ \quad U_{II} = \text{SQRT}(U_{IJ}) \\ \text{Else} \end{array}}$$

$$U_{IJ} = \frac{U_{IJ}}{U_{II}}$$

Endif

2 continue

1 continue

Version 5 Parallel Vector " Skyline" Choleski Code

"Exactly" same as version 4 , except :

$U_{KI} \longrightarrow$ A[MAXA(I)+I-K]

$U_{KJ} \longrightarrow$ A[MAXA(J)+J-K]

$U_{IJ} \longrightarrow$ A[MAXA(J)+J-I]

$S_{IJ} \longrightarrow$ A[MAXA(J)+J-I]

$U_{II} \longrightarrow$ A[MAXA(I)]

$U_{II} \longrightarrow$ A[MAXA(I)]

Note: In actual coding, the decomposed matrix U will overwrite the original stiffness matrix S. For clarity, however, these 2 matrices have been shown under different names

Forward Substitution

To solve $[U]^T\{y\} = \{F\}$ for $\{y\}$

Example :

$$\begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$U_{11}y_1 = F_1 \longrightarrow y_1 = F_1/U_{11}$$

$$U_{12}y_1 + U_{22}y_2 = F_2 \longrightarrow y_2 = F_2 - U_{12}\, y_1/U_{22}$$

multipliers

Similarly $$y_3 = \frac{F_3 - U_{13}\cdot y_1 - U_{23}\cdot y_2}{U_{33}}$$

In general : $$y_j = \frac{F_j - \sum_{i=1}^{j-1} u_{ij}\cdot y_i}{U_{jj}}$$

<u>Version 1</u> Basic scheme

```
        DO 1 J = 1, NEQ
        Sum1 = 0
        DO 2 I = 1, J-1
2       Sum1 = Sum1 + U(I,J)*y(I)
        y(J) = (y(J) – Sum1)/U(J,J)
1       continue
```

<u>Version 2</u> Skyline scheme

```
        DO 1 J= 1, NEQ
        Sum1= 0
c…      DO 2 I = Top row of column J, next to diagonal term, +1
        DO 2 I = colh(J), 1 , -1
2       Sum1 = Sum1 + U[MAXA(J)+I]* y (J-I)
1       y(J) = (y(J) – Sum1)/U(MAXA (J))
```

$$
\begin{array}{cc}
 & \begin{array}{cccccc} & & & \begin{array}{c} J^{\text{th}} \\ \text{Column} \\ \downarrow \end{array} & & \end{array} \\
U = & \begin{bmatrix} x & x & 0 & x & 0 & 0 \\ & x & x & x & x & 0 \\ & & x & x & x & 0 \\ x & x & x & x & x & x \\ \text{sym} & & & & x & x \\ & & & & & x \end{bmatrix}
\end{array}
$$

Backward Substitution

To solve $[U]\{Z\} = \{y\}$

Example:

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

$U_{44} Z_4 = y_4$, hence $Z_4 = \dfrac{y_4}{U_{44}}$

$U_{33} Z_3 + U_{34} Z_4 = y_3$, hence $Z_3 = \dfrac{y_3 - U_{34} Z_4}{U_{33}}$

Similarly: $Z_2 = \dfrac{y_2 - U_{23} Z_3 - U_{24} Z_4}{U_{22}}$

$$Z_1 = \frac{y_1 - U_{12} \cdot Z_2 - U_{13} \cdot Z_3 - U_{14} \cdot Z_4}{U_{11}}$$

In general: $$Z_j = \frac{y_j - \sum_{i=j+1}^{NEQ} U_{ji} Z_i}{U_{jj}}$$

Note

Once Z_4 is known , we can update the right hand side vector $\{y\}$ as following :

$y_3 = y_3 - U_{34} Z_4$

$y_2 = y_2 - U_{24} Z_4$

$y_1 = y_1 - U_{14} Z_4$

Version 1 Basic Scheme

In practice, the solution vector {Z} will overwrite the right hand side vector {y}

```
      DO 1 J = NEQ , 1, -1    (say, J = 4)
      Y(J) = y(J) / U(J,J)

      Do 2 I = J-1, top row of column J, -1

2     y(I) = y(I) – U(I,J)* y(J)
1     continue
```

Version 2 Loop Unrolling Scheme

```
      Do 1 J = NEQ , 1 , -2
      y(J) = y(J)/U(J,J)
      y(J-1) = [ y(J-1)- U(J-1,J) * y(J)] / U(J-1,J-1)
      Do 2 I = J-colh(J), J-2, +1
C…   Do 2 I = J-1, top row of column J, -1

2     y(I) = y(I) –U(I,J) *y(J) – U(I,J-1) * y(J-1)

1     continue
```

A.3 LDL^T Algorithm (or U^TDU)

$$[K] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 \\ 0 & \frac{-2}{3} & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right] \cdot \begin{pmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 0 & -1 & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

Notes

(1) LDL^T is essentially the same as the 2nd version (see A.2) of Gauss

(2) Computer implementation of LDL^T

```
      Do 11 I = 1, N
       Do 22 K = 1, I-1
        xmult=u()/D(I)=u(K,I)/u(K,K)
        Do 33 J = I, N (or I+Irowlength)
         u(I,J) = u(I,J)-xmult*Kth Row
33      Continue
        u(K,I) = xmult
22     Continue
11    Continue
```

u(K,J)

For I = 1 ➔ Temporary no change in 1st row
For I = 2 , Hence K = 1 ➔ 1

$$xmult = \frac{u(1,2)}{u_{11}} = \frac{-1}{2}$$

Loop 33 $\begin{cases} u_{2,2} = u_{2,2} - (xmult)(u_{1,2}) = 2-(-1/2)(-1) = 3/2 \\ u_{2,3} = u_{2,3} - (xmult)(u_{1,3}) = -1-(-1/2)(0) = -1 \end{cases}$

u(1,2) = xmult = -1/2

For I=3 , Hence K= 1 ➔ 2

$$xmult = \frac{u(1,3)}{u_{1,1}} = \frac{0}{2} = 0$$

Loop 33 $\{ \; u_{3,3} = u_{3,3} - (xmult=0)*(u_{1,3}=0) = 1$

u(1,3)=0

Now K = 2

$$xmult = \frac{u(2,3)}{u_{2,2}} = \frac{-1}{\frac{3}{2}} = \frac{-2}{3}$$

Loop 33 $\{ \; u_{3,3} = u_{3,3} - (xmult = -2/3)(u_{2,3} = -1) = 1/3$

u(2,3) = xmult = -2/3 ----------------- QED

Hence:

$$U = \begin{pmatrix} 2 & \frac{-1}{2} & 0 \\ & \frac{3}{2} & \frac{-2}{3} \\ & & \frac{1}{3} \end{pmatrix}$$

[U] matrix, with $U_{ii} = 1$

$[L]^T$ matrix, with $L_{ii} = 1$

[D]

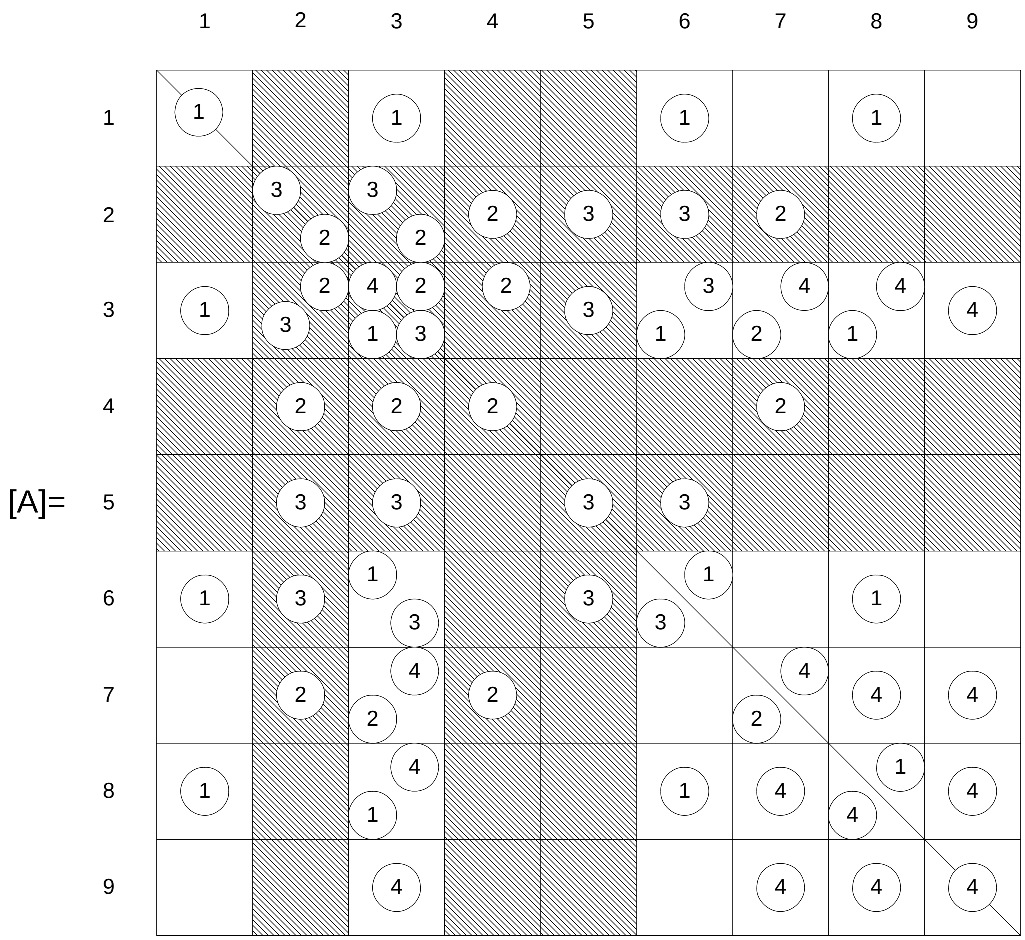


Figure 2 : Total (stiffness) Matrix

IA = 1, 4, 9, 15, 16, 17, 18, 20, 21, 21

JA = 3, 8, 6, 7, 3, 4, 5, 6, 8, 6, 7, 4, 5, 9, 7, 6, 8, 9, 8, 9

Could be unordered

How to impose (Dirichlet) boundary conditions

Assuming $[A]\vec{x} = \vec{b}$, ndof = 4 and with Dirichlet boundary conditions

$x_2 = k_2$ and $x_3 = k_3$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 = k_2 \\ X_3 = k_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \iff \begin{pmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A_{41} & 0 & 0 & A_{44} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} b_1 - A_{12}\cdot k_2 - A_{13}\cdot k_3 \\ k_2 \\ k_3 \\ b_4 - A_{42}\cdot k_2 - A_{43}\cdot k_3 \end{pmatrix}$$

Note: After all x_i are found, reactions

$$R_2 = \sum_{j=1}^{4} A_{2j} \cdot X_j \qquad \text{and} \qquad R_3 = \sum_{j=1}^{4} A_{3j} \cdot X_j$$