1 A Review of Basic Finite Element Procedures

1.2 Numerical Techniques For Solving Ordinary Differential Equations (ODE)

sections.

$$EI\frac{d^2y}{dx^2} = \frac{\omega x(L-x)}{2} \tag{1.1}$$

with the following boundary conditions:

$$y(@\ x=0) = 0 = y(@\ x=L) \tag{1.2}$$

The above equations (1.1-1.2) represent a simply supported beam, subjected to a uniform load applied throughout the beam, as shown in Figure 1.1.

To facilitate the derivation and discussion of the Galerkin method, the ODE given in Eq.(1.1) can be re-casted in the following general form

$$Ly=f \tag{1.3}$$

L and f in Eq.(1.3) represent the “mathematical operator”, and “forcing” function, respectively. Within the context of Eq.(1.1), the “mathematical operator” L, in this case, can be defined as

$$L \equiv EI\frac{d^2}{dx^2}(\ ) \tag{1.4}$$

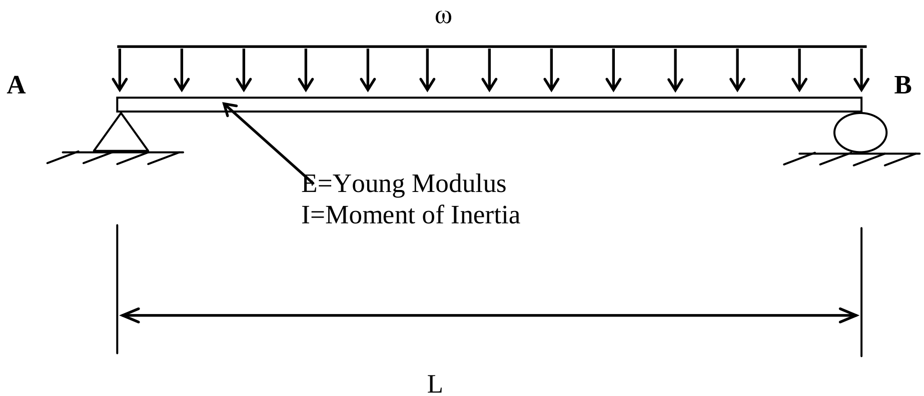


Figure 1.1 A SimplySupported Beam Under Uniformly Applied Load

$$Ly = EI\frac{d^2(y)}{dx^2} \text{ and } f \equiv \frac{w\ x\ (L-x)}{2} \tag{1.5}$$

the virtual work can be computed as

$$\delta w = \iiint_v (f)*\delta y\, dv \equiv \iiint_v (L\ y)*\delta y\, dv \tag{1.6}$$

In Eq.(1.6), δy represents the "virtual" displacement, which is consistent with (or satisfied by) the "geometric" boundary conditions (at the supports at joints A and B of Figure 1.1, for this example).

$$\tilde{y}(x) = \sum_{i=1}^{N} a_i \phi_i(x) \tag{1.7}$$

where a_i are the unknown constant(s), and $\phi_i(x)$ are selected functions such that all "geometric boundary conditions" (such as given in Eq.(1.2)) are satisfied.

Substituting $\tilde{y}(x)$ from Eq.(1.7) into Eq.(1.6), one obtains:

$$\iiint_v (f)\delta y\, dv \neq \iiint_v (L\tilde{y})\, \delta y\, dv \tag{1.8}$$

However, we can adjust the values of a_i (for $\tilde{y}$), such that Eq.(1.8) will be satisfied, hence:

$$\iiint_v (f)\delta y\, dv \approx \iiint_v (L\tilde{y})\, \delta y\, dv \tag{1.9}$$

or

$$\iiint_v (L\tilde{y} - f)\delta y\, dv = 0 \tag{1.10}$$

Based upon the requirements placed on the virtual displacement δy, one may take the following selection:

$$\delta y = \phi_i(x) \tag{1.11}$$

Thus, Eq.(1.10) becomes the "Galerkin" equations, and is given as

$$\iiint_v \underbrace{(L\tilde{y} - f)}_{\substack{\text{Residual} \\ \text{or error}}} \quad \underbrace{\phi_i}_{\substack{\text{Weighting} \\ \text{Function}}} \quad dv = 0 \tag{1.12}$$

Thus, Eq.(1.12) states that the <u>summation</u> of the <u>weighting residual</u> is set to be zero. Substituting Eq.(1.7) into Eq.(1.12), one gets:

$$\iiint_v [L(\sum_{i=1}^{N} a_i \phi_i(x)) - f\,]\,\phi_i\ dv = 0 \tag{1.13}$$

Eq.(1.13) will provide i=1,2,…,N equations, which can be solved simultaneously for obtaining a_i unknowns.

For the example provided in Figure 1.1, the “exact” solution y(x) can be easily obtained as

$$y(x) = \left(\frac{1}{EI}\right) * \left(\frac{2\omega L x^3 - \omega x^4 - \omega L^3 x}{24}\right) \tag{1.14}$$

Assuming a 2-term approximated solution for $\tilde{y}(x)$ is seeked, then (from Eq.(1.7)):

$$\tilde{y} = (A_1 + A_2 x) * \phi_1(x) \tag{1.15}$$

or

$$\tilde{y} = A_1\phi_1(x) + A_2 x \phi_1(x) \tag{1.16}$$

or

$$\tilde{y} = A_1\phi_1(x) + A_2\phi_2(x) \tag{1.17}$$

where

$$\phi_2(x) = x * \phi_1(x) \tag{1.18}$$

Based on the given geometric boundary conditions, given by Eq.(1.2), the function $\phi_1(x)$ can be chosen as:

$$\phi_1(x) = \sin\left(\frac{\pi x}{L}\right) \tag{1.19}$$

Substituting Eqs.(1.18-1.19) into Eq.(1.17), one has:

$$\tilde{y} = A_1 * \sin\left(\frac{\pi x}{L}\right) + A_2 * x \sin\left(\frac{\pi x}{L}\right) \tag{1.20}$$

Substituting the given differential equation (1.1) into the approximated Galerkin Eq.(1.12), one obtains:

$$\int_{x=0}^{L} [EI\tilde{y}'' - \frac{\omega x(L-x)}{2}] * \phi_i(x) dx = 0 \tag{1.21}$$

For i=1, one obtains from Eq.(1.21):

$$\int_{x=0}^{L} [EI\tilde{y}'' - \frac{\omega x(L-x)}{2}] * [\phi_1(x) = \sin(\frac{\pi x}{L})] dx = 0 \tag{1.22}$$

For i=2 one obtains from Eq.(1.21):

$$\int_{x=0}^{L} [EI\tilde{y}'' - \frac{\omega x(L-x)}{2}] * [\phi_2(x) = x\sin(\frac{\pi x}{L})] dx = 0 \tag{1.23}$$

Substituting Eq.(1.20) into Eqs.(1.22-1.23), and performing the integrations using MATLAB, one obtains the following 2 equations:

$$[2EIA_1 * \pi^5 + EIA_2 * \pi^5 * L + 8\omega L^4] = 0 \tag{1.24}$$

$$[3EIA_1 * \pi^5 + 3EIA_2 * \pi^3 * L + 2EIA_2 * \pi^5 * L + 12\omega L^4] = 0 \tag{1.25}$$

or, in the matrix notations:

$$(EI) * \begin{bmatrix} 2\pi^5 & \pi^5 L \\ 3\pi^5 & 3\pi^3 L + 2\pi^5 L \end{bmatrix} * \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} -8\omega L^4 \\ -12\omega L^4 \end{Bmatrix} \tag{1.26}$$

Using MATLAB, the solution of Eq.(1.26) can be given as:

$$A_1 = \frac{-4\omega L^4}{\pi^5 EI} = \frac{(-4)*(5)\omega L^4}{(5)\pi^5 EI} = \frac{-5\omega L^4}{(\frac{5}{4})\pi^5 \ EI} = \frac{-5\omega L^4}{382.523 \ EI} \tag{1.27}$$

$$A_2 = 0 \tag{1.28}$$

Thus, the approximated solution $\tilde{y}(x)$, from Eq.(1.20), becomes:

$$\tilde{y}(x) = \left(\frac{-5\omega L^4}{382.523EI} \right) * \sin(\frac{\pi x}{L}) \tag{1.29}$$

At $x = \frac{L}{2}$, the exact solution is given by Eq.(1.14):

$$y = \frac{-5\omega L^4}{384EI} \tag{1.30}$$

At $x = \frac{L}{2}$, the approximated solution is given by Eq.(1.29):

$$\tilde{y} = \frac{-5\omega L^4}{382.523EI} \tag{1.31}$$

Remarks

(a) The selected function $\phi_1(x)$ can also be easily selected as a polynomial, which also satisfies the geometrical boundary conditions (see Eq.(1.2))

$$\phi_1(x) = (x-0)(x-L) \tag{1.32}$$

and, therefore we have:

$$\tilde{y}(x) = (A_1 + A_2 x) * \phi_1(x) \tag{1.33}$$

or

$$\tilde{y}(x) = A_1\phi_1(x) + A_2\phi_2(x) \tag{1.34}$$

where

$$\phi_2(x) = x\phi_1(x) \tag{1.35}$$

(b) If the function $\phi_1(x)$ has to satisfy the following "hypothetical" boundary conditions,

$$y(@\, x = 0) = 0 \tag{1.36}$$

$$y'(@\, x = 0) = 0 \tag{1.37}$$

$$y(@\, x = L) = 0 \tag{1.38}$$

$$y''(@\, x = 0) = 0 \tag{1.39}$$

then a possible candidate for $\phi_1(x)$ can be chosen as:

$$\phi_1(x) = (x-0)^2 (x-L)^3 \tag{1.40}$$

The first and second terms of Eq.(1.40) are raised to the power 2 and 3 in order to satisfy the "slope" and "curvature" boundary conditions, as indicated in Eq.(1.37), and Eq.(1.39), respectively.

1.3 Identifying the "Geometric" versus "Natural" Boundary Conditions

Let's consider the following "beam" equation:

$$EIy'''' = \omega(x) \tag{1.45}$$

Since the highest order of derivatives involved in the ODE (1.45) is four, one sets:

$$2n = 4 \tag{1.46}$$

hence

$$n - 1 = 1 \tag{1.47}$$

Based on Eq.(1.47), one may conclude that the function itself (=y), and all of its derivatives up to the order "n-1" (=1, in this case, such as y') are the "geometrical" boundary conditions. The "natural" boundary conditions (if any) will involve with derivatives of the order $n(=2), n+1(=3),...$ and higher (such as $y'', y''', \ldots$).

1.4 The Weak Formulations

Let's consider the following differential equation

$$-\frac{d}{dx}\left[a(x)\frac{dy}{dx}\right] = b(x) \quad \text{for} \quad 0 \le x \le L \tag{1.48}$$

$$y(@\, x = 0) = y_0 \tag{1.49}$$

$$\left(a\frac{dy}{dx}\right)_{@\, x=L} = Q_0 \tag{1.50}$$

$$y(x) \approx \tilde{y}(x) = \sum_{i=1}^{N} A_i \phi_i(x) + \phi_0(x) \tag{1.51}$$

Based on the discussions in Section 1.3, one obtains from Eq.(1.48):

2n=2 (=the highest order of derivative) (1.52)

Hence:

$$n-1=0 \tag{1.53}$$

Thus, Eq.(1.49) and Eq.(1.50) represent the "geometrical" and "natural" boundary conditions, respectively.

The non-homogeneous "geometrical" (or "essential") boundary conditions can be satisfied by the function $\phi_0(x)$, such as $\phi_0(@\, x_0) = y_0$. The functions $\phi_i(x)$ are required to satisfy the homogeneous form of the same boundary condition $\phi_i(@\, x_0) = 0$.

If all specified "geometrical" boundary conditions are homogeneous (for example, y_0=0), then $\phi_0(x)$ is taken to be zero, and $\phi_i(x)$ must still satisfy the same boundary conditions (for example, $\phi_i(@\, x_0) = 0$).

The "weak formulation", if it exists, basically involves the following 3 steps:

Step 1:

$$\int_0^L \Big[\underbrace{-\frac{d}{dx}\left\{a(x)\frac{dy}{dx}\right\}}_{dv} - b(x) \Big] * \underbrace{W_i}_{u}(x)dx = 0 \tag{1.54}$$

Step 2:

$$-\left[W_i(x) * a(x)\frac{dy}{dx}\right]_0^L + \int_0^L [a(x)\frac{dy}{dx} * \frac{dW_i(x)}{dx} - b(x) * W_i(x)]\,dx = 0 \tag{1.55}$$

$$a(x)\frac{dy}{dx} * n_x \equiv Q \equiv \text{"secondary" variable (=Heat, for example)} \tag{1.56}$$

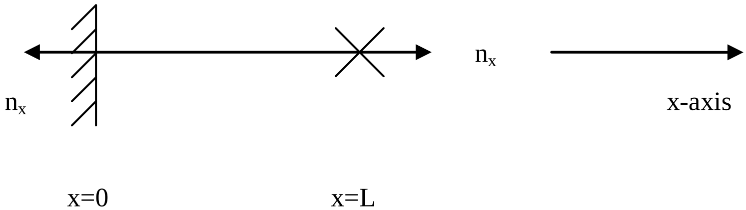


Figure 1.2 Normal (=n_x) to the boundary of the beam

$$0 = -[W_i(x) * a(x)\frac{dy}{dx}]^{@x=L} + [W_i(x) * a(x)\frac{dy}{dx}]_{@x=0}$$
$$+ \int_0^L [a(x)\frac{dy}{dx}\frac{dW_i(x)}{dx} - b(x)W_i(x)]dx$$

$$0 = -[W_i(x) * a(x)\frac{dy}{dx} * n_x]^{@x=L} - [W_i(x) * a(x)\frac{dy}{dx} * n_x]_{@x=0}$$
$$+ \int_0^L [a(x)\frac{dy}{dx}\frac{dW_i(x)}{dx} - b(x)W_i(x)]dx$$

or:

$$0 = \int_0^L [a(x)\frac{dy}{dx}\frac{dW_i(x)}{dx} - b(x)W_i(x)]dx - [W_i(x) * Q]^{@x=L} + [W_i(x) * Q]_{@x=0} \tag{1.57}$$

It should be noted that in the form of Eq.(1.54), the "primary" variable y(x) is required to be <u>twice</u> differentiable, but only <u>once</u> in Eq.(1.57).

<u>Step 3:</u>

In this last step of the weak formulation, the actual boundary conditions of the problem are imposed. Since the weighting function $W_i(x)$ is required to satisfy the homogeneous form of the specified geometrical (or essential) boundary conditions, hence:

$$W_i(@x = 0) = 0 \ ; \ \text{because } y(@x=0)=y_0 \tag{1.58}$$

Eq.(1.57) will reduce to:

$$\int_0^L [a(x)\frac{dy}{dx} - b(x) * W_i(x)]\, dx - W_i(@x = L) * Q(@x = L)$$

or, using the notation introduced in Eq.(1.50), one has:

$$\int_0^L [a(x)\frac{dy}{dx} - b(x) * W_i(x)]\, dx - W_i(@L) * Q_0 \tag{1.59}$$

Eq.(1.59) is the weak form equivalent to the differential equation (1.48), and the natural boundary condition equation (1.50).

1.5 Flowcharts For Statics Finite Element Analysis

$$\{f(x_i)\}_{3x1} = [N(x_i)]_{3xn} * \{r'\}_{nx1} \tag{1.60}$$

$$\{\varepsilon\} = \frac{\partial}{\partial x_i}[N(x_i)] * \{r'\} \tag{1.61}$$

or

$$\{\varepsilon\} = [B(x_i)] * \{r'\} \tag{1.62}$$

where:

$$[B(x_i)] \equiv \frac{\partial}{\partial x_i}[N(x_i)] \tag{1.63}$$

The <u>internal</u> virtual work can be equated with the <u>external</u> virtual work, therefore

$$\int_V \delta\,\varepsilon^T * \sigma\, dV = \{\varepsilon\, r'\}^T * \{p'\} \tag{1.64}$$

$$\{\sigma\} = [D]\{\varepsilon\} \tag{1.65}$$

$$\sigma_{xx} = E\,\varepsilon_{xx} \tag{1.66}$$

From Eq.(1.62), one obtains:

$$\{\varepsilon\}^T = \{r'\}^T [B]^T \tag{1.67}$$

$$\{\delta\varepsilon\}^T = \{\delta r'\}^T [B]^T \tag{1.68}$$

$$\int_V \{\delta r'\}^T [B]^T * [D]\{\varepsilon = B\, r'\}\, dV - \{\delta r'\}^T \{p'\} = 0 \tag{1.69}$$

$$\{\delta r'\}^T * \left[\int_V [B]^T [D][B] dV * \{r'\} - \{p'\} \right] = 0 \tag{1.70}$$

$$[k'] * \{r'\} - \{p'\} = \{0\} \tag{1.71}$$

$$[k'] \equiv \int_V [B]^T [D][B] dV \tag{1.72}$$

Since the element local coordinate axis, in general will NOT coincide with the system global coordinate axis, the following transformation need be done (see Figure 1.3):

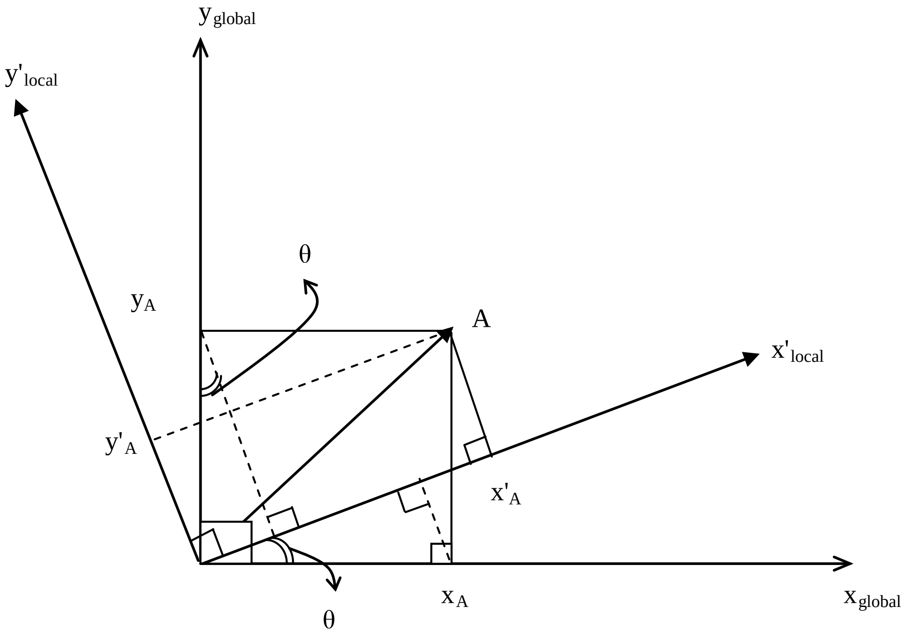


Figure 1.3 Local-Global Coordinate Transformation

$$\{r'\} = \begin{Bmatrix} x'_A \\ y'_A \end{Bmatrix} = \begin{Bmatrix} x_A \cos(\theta) + y_A \sin(\theta) \\ y_A \cos(\theta) - x_A \sin(\theta) \end{Bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{Bmatrix} x_A \\ y_A \end{Bmatrix} \tag{1.73}$$

$$\{r'\} = [\lambda] \begin{Bmatrix} x_A \\ y_A \end{Bmatrix} \equiv [\lambda]\{r\} \tag{1.74}$$

$$[\lambda] \equiv \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} ; \text{ for 2-D problems} \tag{1.75}$$

$$\{p'\} = [\lambda]\{p\} \tag{1.76}$$

$$[k'] * [\lambda]\{r\} = [\lambda]\{p\} \tag{1.77}$$

$$[\lambda]^T [k'][\lambda] * \{r\} = [\lambda]^T [\lambda]\{p\} \tag{1.78}$$

$$[\lambda]^T [k'][\lambda] * \{r\} = \{p\} \tag{1.79}$$

$$[k] * \{r\} = \{p\} \tag{1.80}$$

$$[k] = [\lambda]^T [k'][\lambda] \equiv [k^{(e)}] = \text{element stiffness in global coordinate reference} \tag{1.81}$$

$$[K] * \{R\} = \{P\} \tag{1.82}$$

$$[K] = \sum_{e=1}^{NEL} [k^{(e)}] \tag{1.83}$$

$\{P\} = \sum_{e=1}^{NEL} \{p^{(e)}\}$; and $\{p^{(e)}\}$ is the right-hand-side of Eq.(1.80)

$\{R\}$ = system unknown displacement vector

$$\{\Lambda\} = [\Phi]^T [M]\{R^{(0)}\} \equiv \{\Lambda^{(0)}\} \tag{1.126}$$

1.8 One-Dimensional Rod Finite Element Procedures

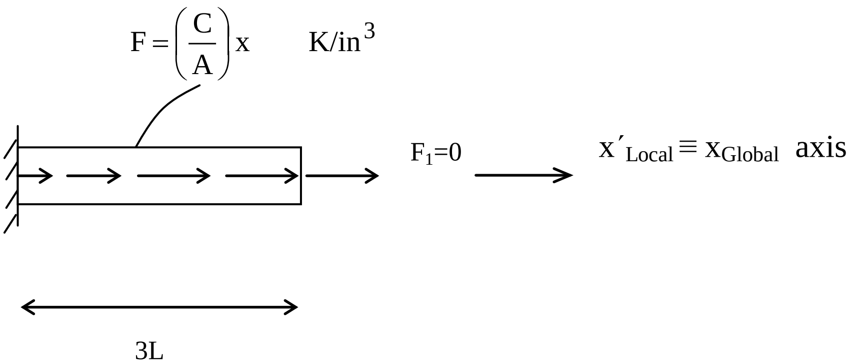


Figure 1.4 Axially Loaded Rod

$$-EA\frac{\partial^2 u}{\partial x^2} = q(x) \tag{1.127}$$

$$u(@\,x = 0) = u_0 = 0 \tag{1.128}$$

$$EA\left.\frac{\partial u}{\partial x}\right|_{@\,x=3L} = F_1 \quad (\equiv \text{Axial Force}) \tag{1.129}$$

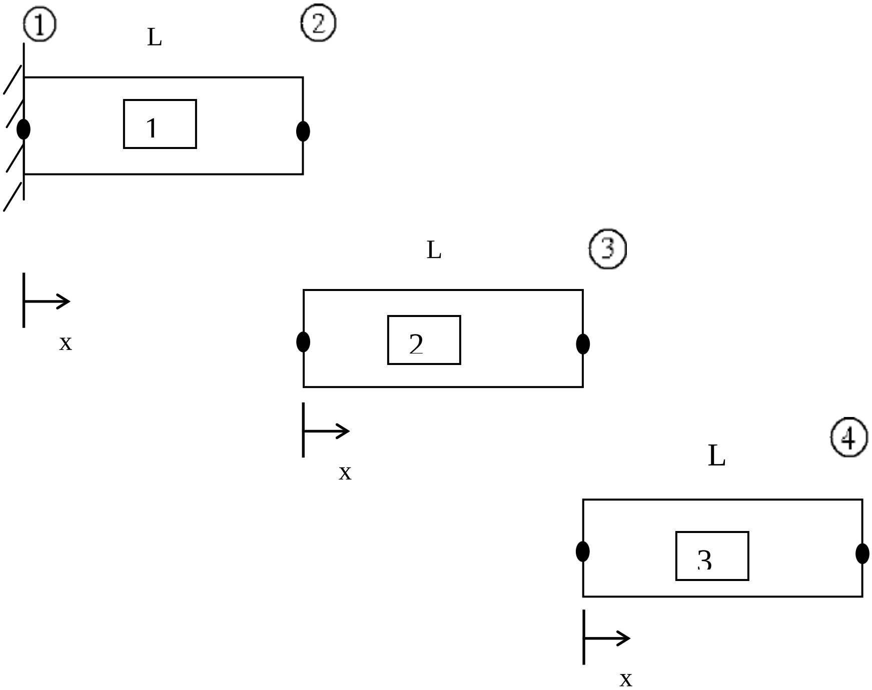


Figure 1.5 An Axially Loaded Rod With 3 Finite Elements

1.8.1 1-D Rod Element Stiffness Matrix

$$u(x) = a_1 + a_2 x = [1, x] * \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{1.130}$$

$$\text{At } x = 0;\ u = u_1 = a_1 + a_2(x = 0) \tag{1.131}$$
$$\text{At } x = L;\ u = u_2 = a_1 + a_2(x = L) \tag{1.132}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} * \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{1.133}$$

$$\{u\} = [A]\{a\} \tag{1.134}$$

$$\{u\} \equiv \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{1.135}$$

$$[A] \equiv \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \tag{1.136}$$

$$\{a\} \equiv \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \quad (1.137)$$

$$\{a\} = [A]^{-1} * \{u\} \quad (1.138)$$

$$[A]^{-1} = \left(\frac{1}{L}\right)\begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \quad (1.139)$$

$$u(x) = [1, x] * [A]^{-1} * \{u\} \quad (1.140)$$

$$u(x) = [1, x] * \begin{bmatrix} 1 & 0 \\ \frac{-1}{L} & \frac{1}{L} \end{bmatrix} * \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (1.141)$$

Eq.(1.141) has exactly the same form as described earlier by Eq.(1.160), where

$$\{f(x_i)\} \equiv u(x) \quad (1.142)$$

$$[N(x)] \equiv [1, x] * \begin{bmatrix} 1 & 0 \\ \frac{-1}{L} & \frac{1}{L} \end{bmatrix} = \{1 - \frac{x}{L}, \frac{x}{L}\} \quad (1.143)$$

$$\{r'\} \equiv \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (1.144)$$

The shape (or interpolation) functions $[N(x)] = [\phi_1^e(x) = 1 - \frac{x}{L}; \phi_2^e(x) = \frac{x}{L}]$, shown in Eq.(1.143), have the following properties :

$$\phi_i^e(@ x_j) = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \quad (1.143a)$$

$$\sum_{i=1}^{n} \phi_i(x) = 1, \quad \text{therefore} \quad \sum_{i=1}^{n} \frac{d\phi_i(x)}{dx} = 0 \quad (1.143b)$$

where $n \equiv$ the number of dof per element

$$[B(x)] = \frac{\partial}{\partial x}[N(x)] = \left\{\frac{-1}{L}, \frac{1}{L}\right\} \quad (1.145)$$

$$[k'] = \int_0^L [B]^T (D)[B]dx \quad (1.146)$$

$$[k'] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} (D) \left[-\frac{1}{L}, \frac{1}{L} \right] dx \tag{1.147}$$

$$[k'] = \frac{(D)}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(AE)}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.148}$$

1.8.2 Distributed Loads and Equivalent Joint Loads

The given traction force per unit volume (see Figure 1.4)

$$F = \frac{cx}{A} \quad K/in^3 \tag{1.156}$$

can be converted into the distributed load as

$$q = F*A = cx \quad K/in \tag{1.157}$$

The work done by the external loads acting on finite element rod #1 can be computed as:

$$W = \iiint_V F*u*dV = \int_{x=0}^{L} FuAdx \tag{1.158}$$

$$W = \int_0^L (cx)*([N]\{r'\})dx \tag{1.159}$$

$$W = \int_0^L (cx)*\{r'\}^T [N]^T dx = \{r'\}^T \int_0^L q*[N]^T dx \tag{1.160}$$

$$W = \{r'\}^T * \int_0^L (cx) \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} dx \tag{1.161}$$

$$\text{Let} \quad \{F_{equiv}\} \equiv \int_0^L (cx) \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} dx \tag{1.162}$$

$$\{F^1{}_{equiv}\} = \frac{cL^2}{6} * \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \tag{1.163}$$

$$W = \{r'\}^T * \{F_{equiv}\} \tag{1.164}$$

$$\{F_{equiv}\} \equiv \int_0^L (q)*[N(x)]^T dx \qquad (1.165)$$

Similarly, for finite element rod #2 and #3, one obtains (see Fig. 1.4)

$$\{F^2{}_{equiv}\} \equiv \int_0^L c(L+x)[N]^T dx \qquad (1.166)$$

$$\{F^2_{equiv}\} = \left(\frac{cL^2}{6}\right)\begin{Bmatrix} 4 \\ 5 \end{Bmatrix} \qquad (1.167)$$

$$\{F^3{}_{equiv}\} \equiv \int_0^L c(2L+x)[N]^T dx = \left(\frac{cL^2}{6}\right)\begin{Bmatrix} 7 \\ 8 \end{Bmatrix} \qquad (1.168)$$

1.8.3 Finite Element Assembly Procedures

Thus, in this case the transformation matrix $[\lambda]$ (see Eq.1.75) becomes an identity matrix, and therefore from Eq.(1.81), one gets:

$$[k] = [k']$$

$$[K]_{4x4} = \sum_{e=1}^{NEL=3} [k^{(e)}] \qquad (1.169)$$

$$[K]_{4x4} = \left(\frac{AE}{L}\right)*\left(\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\right)$$

$$[K]_{4x4} = \left(\frac{AE}{L}\right)*\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad (1.170)$$

Similarly, the system nodal load vector can be assembled from its elements' contributions:

$$\{P\}_{4x1} = \sum_{e=1}^{NEL=3} \{p^{(e)}\} \equiv \sum_{e=1}^{NEL=3} \{F^{(e)}_{Equiv.}\} \qquad (1.171)$$

$$\{P\}_{4x1} = \left(\frac{cL^2}{6}\right) * \left(\begin{Bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 4 \\ 5 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 7 \\ 8 \end{Bmatrix} \right) \tag{1.172}$$

$$\{P\}_{4x1} = \left(\frac{cL^2}{6}\right) * \begin{Bmatrix} 1 \\ 6 \\ 12 \\ 8 \end{Bmatrix} \tag{1.173}$$

$$\left(\frac{AE}{L}\right) \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} * \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \left(\frac{cL^2}{6}\right) \begin{Bmatrix} 1 \\ 6 \\ 12 \\ 8 \end{Bmatrix} \tag{1.174}$$

$$[K]_{4x4} * \{D\}_{4x1} = \{P\}_{4x1} \tag{1.175}$$

1.8.4 Imposing The Boundary Conditions

To make the discussions more general, let's assume that the boundary condition is prescribed at node 1 as"

$$u_1 = \alpha_1 \ \text{(where } \alpha_1 = \text{known value)} \tag{1.176}$$

Let $F_{unknown1}$ be defined as the unknown axial "reaction" force at the supported node 1, then Eq.(1.174), after imposing the boundary condition(s), can be symbolically expressed as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} * \begin{Bmatrix} u_1 = \alpha_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{unknown1} \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{1.177}$$

Eq.(1.177) is <u>equivalent</u> to the following matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & K_{24} \\ 0 & K_{32} & K_{33} & K_{34} \\ 0 & K_{42} & K_{43} & K_{44} \end{bmatrix} * \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \\ P_2 - K_{21} * \alpha_1 \\ P_3 - K_{31} * \alpha_1 \\ P_4 - K_{41} * \alpha_1 \end{Bmatrix} \tag{1.178}$$

$$[K_{bc}] * \{D\} = \{P_{bc}\} \tag{1.179}$$

1.8.5 Alternative Derivations of System of Equations from Finite Element Equations

From a given ODE, shown in Eq.(1.127), which can be applied for a typical e^{th} element (shown in Figure 1.5), as following :

<u>Step 1</u>: Setting the integral of weighting residual to zero

$$0 = \int_{x_A}^{x_B} w \left[R = -EA \frac{\partial^2 u}{\partial x^2} - q(x) \right] dx \tag{1.179a}$$

<u>Step 2</u>: Integrating by parts once

$$0 = \left[w * (-EA \frac{\partial u}{\partial x}) \right]_{x_A}^{x_B} - \int_{x_A}^{x_B} (-EA \frac{\partial u}{\partial x}) * \frac{\partial w}{\partial x} dx - \int_{x_A}^{x_B} w * q(x)\, dx \tag{1.179b}$$

Let $Q \equiv +EA \frac{\partial u}{\partial x} n_x$ (for the boundary terms) (1.179c)

where n_x has already been used/defined in Eq.(1.56)

$$0 = [-w(@x_B) * Q(@x_B)] - [w(@x_A) * Q(@x_A)] + \int_{x_A}^{x_B} (EA \frac{\partial u}{\partial x}) * \frac{\partial w}{\partial x} dx - \int_{x_A}^{x_B} w * q(x)\, dx \tag{1.179d}$$

Step 3: Imposing "actual" boundary conditions

$$0 = \int_{x_A}^{x_B} (EA\frac{\partial u}{\partial x}) * \frac{\partial w}{\partial x}\, dx - w(x_A) * Q_A - w(x_B) * Q_B - \int_{x_A}^{x_B} w * q(x)\, dx \tag{1.179e}$$

Step 4: Finite Element Equations

$$\text{Let } w \equiv \phi_i^e(x) \tag{1.179f}$$

$$\text{Let } u \equiv \sum_{j=1}^{n} u_j^e\, \phi_j^e(x) \tag{1.179g}$$

$$0 = \int_{x_A}^{x_B} EA\frac{\partial}{\partial x}(\sum_{j=1}^{n} u_j^e\, \phi_j^e) * \frac{\partial \phi_i^e}{\partial x}\, dx - \sum_{j=1}^{n} \phi_i^e(@x_j) * Q_j - \int_{x_A}^{x_B} \phi_i^e * q(x)\, dx \tag{1.179h}$$

$$\sum_{j=1}^{n} \phi_i^e(@x_j) * Q_j = \sum_{j=1}^{n} \delta_{ij} * Q_j \equiv Q_i^e \tag{1.179i}$$

$$0 = \int_{x_A}^{x_B} EA\frac{\partial}{\partial x}(\sum_{j=1}^{n} u_j^e\, \phi_j^e) * \frac{\partial \phi_i^e}{\partial x}\, dx - Q_i^e - \int_{x_A}^{x_B} \phi_i^e * q(x)\, dx \tag{1.179j}$$

$$0 = \left(\sum_{j=1}^{n} \left[k_{ij}^e\right] * u_j^e\right) - f_i^e - Q_i^e \tag{1.179k}$$

where i=1, 2, …, n (= number of dof per element)

$$\text{and } [k_{ij}^e] \equiv \int_{x_A}^{x_B} (EA\frac{\partial \phi_i^e}{\partial x} * \frac{\partial \phi_j^e}{\partial x})\, dx \quad \equiv \quad \text{element “stiffness” matrix} \tag{1.179l}$$

$$\left\{f_i^e\right\} \equiv \int_{x_A}^{x_B} q(x) * \phi_i^e \, dx \equiv \text{"Equivalent" nodal load vector} \qquad (1.179m)$$

$$\{Q_i^e\} \equiv \text{"Applied" nodal load vector} \qquad (1.179n)$$

$$[k^e]*\{u^e\} = \{f^e\} + \{Q^e\} \qquad (1.179o)$$

Remarks :

(1) Referring to Figure 1.5, a typical eth finite element will have :

- 2 equations [see Eq.(1.179k), assuming n=2]
- 4 unknowns (say, for the 2nd finite element), which can be identified as u_2^2, u_3^2, Q_2^2 and Q_3^2

(2) Thus, for the "entire" domain (which contains all 3 finite elements, as shown in Figure 1.5), one has :

- $2\dfrac{equations}{element} * 3\,elements = 6\,equations$
- 12 unknowns, which can be identified as $u_1^1, u_2^1, Q_1^1 \text{ and } Q_2^1, \cdots, u_3^3, u_4^3, Q_3^3, Q_4^3$

(3) The additional (6, in this example) equations can be obtained from :

- System, geometrical boundary condition(s), such as $u_1 = 0$
- Displacement compatibility requirements at "common" nodes (between "adjacent" finite elements), such as :
  $u_2^1 = u_2^2$
  $u_3^2 = u_3^3$
- Applied "nodal" loads, such as :
  $Q_2^1 + Q_2^2 = 0$
  $Q_3^2 + Q_3^3 = 0$
  $Q_4^3 = F_1$ (see Figures 1.4 and 1.5)

1.9 Truss Finite Element Equations

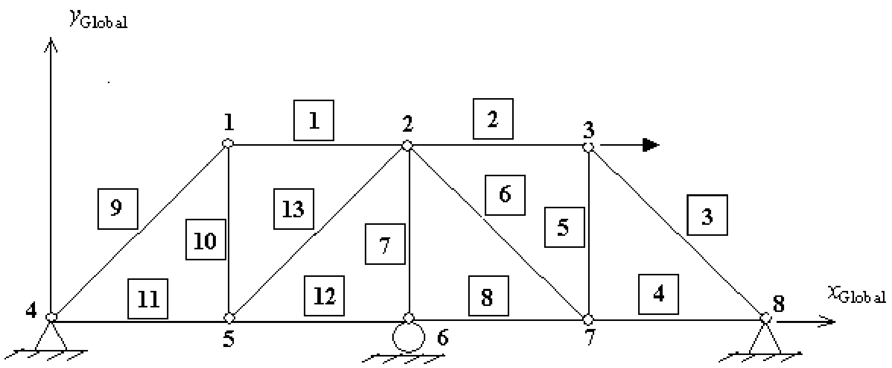


Figure 1.6 A Typical 2-D Truss Structure

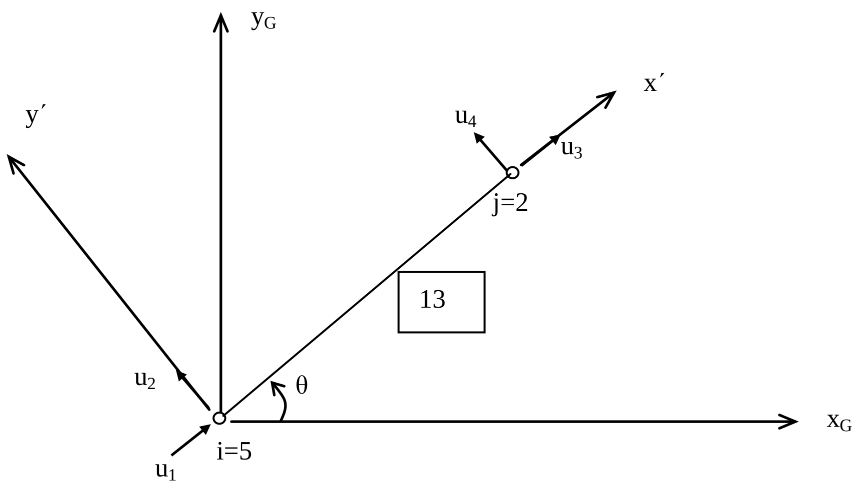


Figure 1.7 A Typical 2-D Truss Member With Its Nodal Displacements

$$[k^{1(e)}] = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.180}$$

The 4x4 global element stiffness matrix of a 2-D truss element can be computed from Eq.(1.81) as:

$$[k^{(e)}]_{4x4} = \begin{bmatrix} [\lambda] & [0] \\ [0] & [\lambda] \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [\lambda] & [0] \\ [0] & [\lambda] \end{bmatrix} \left(\frac{AE}{L}\right) \tag{1.181}$$

1.10 Beam (or Frame) Finite Element Equations

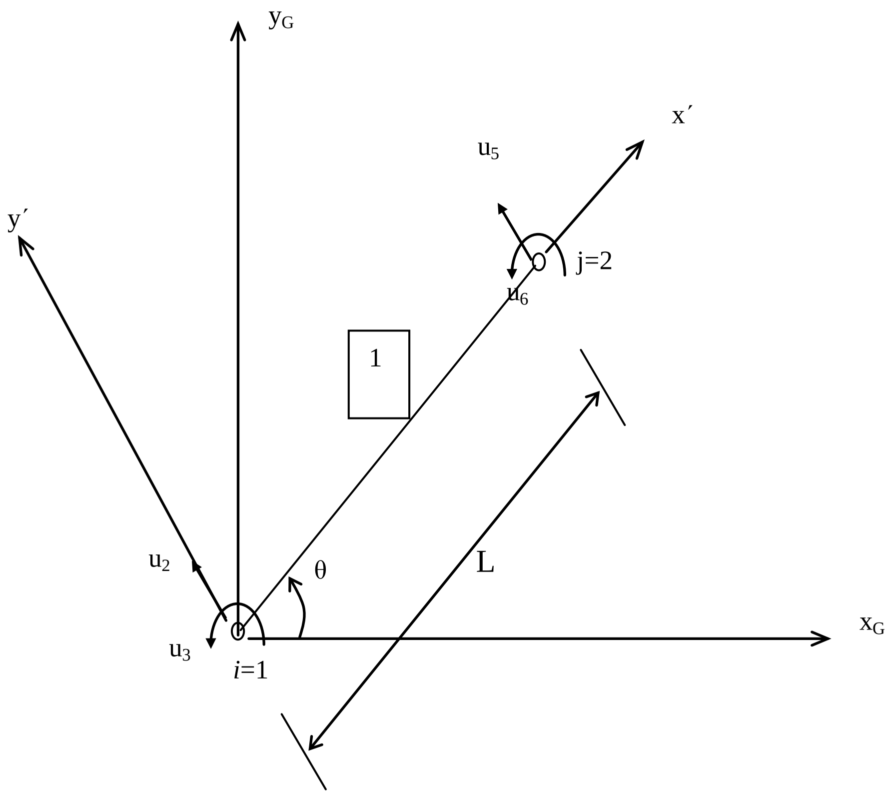


Figure 1.8 A Typical 2-D Beam Element Without Axial dof

$$\frac{d^2}{dx^2}\left(D\frac{d^2\omega}{dx^2}\right) = f(x), \quad \text{for } 0 \le x \le L \tag{1.182}$$

$$[D]_{1x1} = EI \tag{1.183}$$

The transverse displacement field ω(x) within a 4 dof 2-D beam element can be given as:

$$\omega(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 = [1, x, x^2, x^3]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{1.184}$$

$$\omega'(x) = \frac{d\omega}{dx} = a_2 + 2a_3x + 3a_4x^2 = [0,1,2x,3x^2]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{1.185}$$

$$\text{At } x = 0,\ \omega = \omega_1 \tag{1.186}$$

$$\text{At } x = 0,\ \omega' = \left.\frac{d\omega}{dx}\right|_{x=0} = \theta_2 \tag{1.187}$$

$$\text{At } x = L,\ \omega = \omega_3 \tag{1.188}$$

$$\text{At } x = 0,\ \omega' = \left.\frac{d\omega}{dx}\right|_{x=L} = \theta_4 \tag{1.189}$$

$$\begin{Bmatrix} \omega 1 \\ \theta 2 \\ \omega 3 \\ \theta 4 \end{Bmatrix} = \begin{bmatrix} 1,0,0,1 \\ 0,1,0,0 \\ 1, L, L^2, L^3 \\ 0,1,2L,3L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{1.190}$$

$$\{u\}_{4x1} = [A]_{4x4}\{a\}_{4x1} \tag{1.191}$$

$$\{a\} = [A]^{-1}\{u\} \tag{1.192}$$

$$\omega(x) = [1, x, x^2, x^3]_{1x4}[A]^{-1}_{4x4}\{u\}_{4x1} \tag{1.193}$$

$$\omega(x) = [N(x)]_{1x4} * \{u\}_{4x1} \tag{1.194}$$

$$[N(x)]_{1x4} = [1, x, x^2, x^3]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}^{-1} \tag{1.195}$$

$$[N(x)] = \left\{ \frac{L^3 - 3Lx^2 + 2x^3}{L^3}, \frac{x(L^2 - 2Lx + x^2)}{L^2}, \frac{-x^2(-3L + 2x)}{L^3}, \frac{x^2(-L + x)}{L^2} \right\} \tag{1.196}$$

$$\{F_{equiv}\} = \int_0^L (f_0) * [N(x)]^T dx \tag{1.197}$$

$$\{F_{equiv}\} = \begin{Bmatrix} \dfrac{f_0 * L}{2} \\ \dfrac{f_0 * L^2}{12} \\ \dfrac{f_0 * L}{2} \\ \dfrac{-f_0 * L^2}{12} \end{Bmatrix} \tag{1.198}$$

1.11 Weak Formulation for the Beam (or Frame) Finite Element Equations

Step 1

Applying the Galerkin Weighted Residual Integral (see Eq.1.12) into the given beam's differential equation (1.182), one gets

$$0 = \int_0^L W_i \left[\frac{d^2}{dx^2} \left(D \frac{d^2 \omega}{dx^2} \right) - f(x) \right] dx \tag{1.199}$$

Step 2
Upon integrating by parts TWICE, Eq.(1.199) becomes:

$$0=\int_0^L\left[D\frac{d^2W_i}{dx^2}\frac{d^2\omega}{dx^2}-W_i f(x)\right]dx+\left[W_i\frac{d}{dx}\left(D\frac{d^2\omega}{dx^2}\right)-\frac{dW_i}{dx}D\frac{d^2\omega}{dx^2}\right]_0^L \tag{1.200}$$

To simplify the notations, let's define:

$$Q_1^{(e)}\equiv\left[\frac{d}{dx}\left(D\frac{d^2\omega}{dx^2}\right)\right]_{@x=0} \tag{1.201}$$

$$Q_2^{(e)}\equiv\left[D\frac{d^2\omega}{dx^2}\right]_{@x=0} \tag{1.202}$$

$$Q_3^{(e)}\equiv-\left[\frac{d}{dx}\left(D\frac{d^2\omega}{dx^2}\right)\right]_{@x=L} \tag{1.203}$$

$$Q_4^{(e)}\equiv-\left[D\frac{d^2\omega}{dx^2}\right]_{@x=L} \tag{1.204}$$

$$\begin{aligned}0=&\int_0^L\left[D\frac{d^2W_i}{dx^2}\frac{d^2\omega}{dx^2}-W_i f(x)\right]dx-W_i(@x=L)*Q_3^{(e)}+\left(\frac{dW_i}{dx}\right)_{@x=L}*Q_4^{(e)}\\&-W_i(@x=0)*Q_1^{(e)}+\left(\frac{dW_i}{dx}\right)_{@x=0}*Q_2^{(e)}\end{aligned} \tag{1.205}$$

$$0=\int_0^L\left[D\frac{d^2N_i}{dx^2}*\frac{d^2(N_j u_j)}{dx^2}\right]dx-\int_0^L N_i f(x)dx+(-1)^i Q_i^{(e)} \tag{1.206}$$

$$0=\int_0^L\left[D\frac{d^2N_i}{dx^2}*\frac{d^2N_j}{dx^2}\right]dx*\{u_j\}-\int_0^L N_i f(x)dx+(-1)^i Q_i^{(e)} \tag{1.207}$$

$$\sum_{j=1}^{4}\left(\int_0^L\left[D\frac{d^2N_i}{dx^2}*\frac{d^2N_j}{dx^2}\right]dx\right)*\{u_j\}=\int_0^L N_i f(x)dx-(-1)^i Q_i^{(e)} \tag{1.208}$$

$$\sum_{j=1}^{4}[k_{ij}^{(e)}]*\{u_j^{(e)}\}=\{R_i^{(e)}\} \tag{1.209}$$

$$[k_{ij}^{(e)}] \equiv \int_0^L \left[D \frac{d^2 N_i}{dx^2} \frac{d^2 N_j}{dx^2} \right] dx \qquad (1.210)$$

$$u_j^{(e)} = \begin{Bmatrix} \omega_1 \\ \theta_2 \\ \omega_3 \\ \theta_4 \end{Bmatrix} \qquad (1.211)$$

$$R_i^{(e)} = \int_0^L N_i f(x) dx + (-1)^i Q_i^{(e)} \qquad (1.212)$$

$$[k_{ij}^{(e)}] = \frac{2(D = EI)}{L^3} * \begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \qquad (1.213)$$

$$\{R_i^{(e)}\} = \begin{Bmatrix} \frac{f * L}{2} \\ \frac{f * L^2}{12} \\ \frac{f * L}{2} \\ \frac{-f * L^2}{12} \end{Bmatrix} + \begin{Bmatrix} -Q_1^{(e)} \\ Q_2^{(e)} \\ -Q_3^{(e)} \\ Q_4^{(e)} \end{Bmatrix} \qquad (1.214)$$

In general, the equilibrium of the generalized forces at a node between two adjacent elements (e) and (e+1) requires that

$$(-Q_3^{(e)}) + (-Q_1^{(e+1)}) = \text{applied external concentrated force} \qquad (1.215)$$

$$Q_4^{(e)} + Q_2^{(e)} = \text{applied external concentrated moment} \qquad (1.216)$$

1.13 Tetrahedral Finite Element Shape Functions

The governing 3-D Poisson equation can be given as

$$-\frac{\partial}{\partial x}\left(c_1\frac{\partial\omega}{\partial x}\right)-\frac{\partial}{\partial y}\left(c_2\frac{\partial\omega}{\partial y}\right)-\frac{\partial}{\partial z}\left(c_3\frac{\partial\omega}{\partial z}\right)=f \text{ in } \Omega \tag{1.268}$$

with the following geometric boundary condition(s):

$$\omega = \omega_0 \text{ on } \Gamma_1$$

and the natural boundary condition(s)

$$c_1\frac{\partial\omega}{\partial x}n_x + c_2\frac{\partial\omega}{\partial y}n_y + c_3\frac{\partial\omega}{\partial z}n_z = q_0 \text{ on } \Gamma_2 \tag{1.269}$$

where $c_i=c_i(x,y,z)$ and $f=f(x,y,z)$ are given functions on the boundaries Γ_1 and Γ_2, respectively.

The weak formulation can be derived by the familiar 3-step procedures:

Step 1

Setting the weighted residual of the given differential equation to be zero, thus:

$$0=\int_{\Omega^e} W\left[-\frac{\partial}{\partial x}\left(c_1\frac{\partial\omega}{\partial x}\right)-\frac{\partial}{\partial y}\left(c_2\frac{\partial\omega}{\partial y}\right)-\frac{\partial}{\partial z}\left(c_3\frac{\partial\omega}{\partial z}\right)-f\right]d\Omega \tag{1.270}$$

where $d\Omega \equiv dxdydz$ (1.271)

Step 2

Eq.(1.270) can be integrated by parts once, to give:

$$0=\int_{\Gamma^e} W\left[-c_1\frac{\partial\omega}{\partial x}n_x - c_2\frac{\partial\omega}{\partial y}n_y - c_3\frac{\partial\omega}{\partial z}n_z\right] - \int_{\Omega^e}\left[-c_1\frac{\partial\omega}{\partial x}\frac{\partial W}{\partial x} - c_2\frac{\partial\omega}{\partial y}\frac{\partial W}{\partial y} - c_3\frac{\partial\omega}{\partial z}\frac{\partial W}{\partial z} + Wf\right]d\Omega \tag{1.272}$$

Let

$$q_n \equiv c_1\frac{\partial\omega}{\partial x}n_x + c_2\frac{\partial\omega}{\partial y}n_y + c_3\frac{\partial\omega}{\partial z}n_z \tag{1.273}$$

Then, Eq.(1.272) can be re-written as

$$0=\int_{\Omega^e}\left[c_1\frac{\partial\omega}{\partial x}\frac{\partial W}{\partial x} + c_2\frac{\partial\omega}{\partial y}\frac{\partial W}{\partial y} + c_3\frac{\partial\omega}{\partial z}\frac{\partial W}{\partial z} - Wf\right]d\Omega - \oint_{\Gamma^e} W q_n \, d\Gamma \tag{1.274}$$

The primary dependent function ω can be assumed as:

$$\omega = \sum_{j=1}^{n} \omega_j N_j^e(x,y,z) \equiv [N(x,y,z)]_{1xn} * \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \vdots \\ \omega_n \end{Bmatrix}_{nx1} \quad (1.275)$$

In Eq.(1.275), n, ω_j and N_j represent the number of dof per element, element nodal displacements, and element shape functions, respectively.

For a 4-node tetrahedral element (see Figure 1.10) n=4, the assumed field can be given as

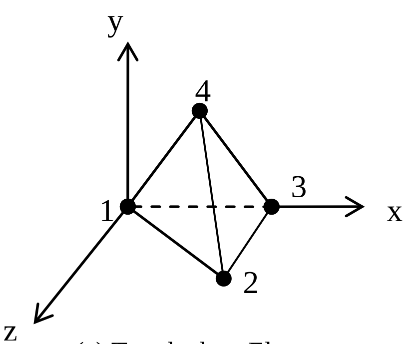
$$\omega(x,y,z) = a_1 + (a_2 x + a_3 y + a_4 z) \quad (1.276)$$

or

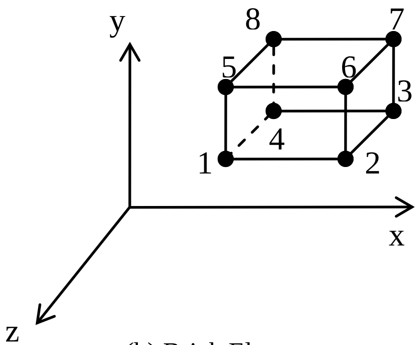
$$\omega(x,y,z) = [1, x, y, z] * \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad (1.277)$$

For an 8-node brick element (see Figure 1.10), n=8, the assumed field can be given as:

$$\omega(x,y,z) = a_1 + (a_2 x + a_3 y + a_4 z) + (a_5 xy + a_6 yz + a_7 zx) + (a_8 xyz) \quad (1.278)$$



(a) Tetrahedron Element



(b) Brick Element

Figure 1.10 Three-Dimensional Solid Elements

The shape functions for the 4-node tetrahedral element can be obtained by the same familiar procedures. The geometric boundary conditions associated with an e^{th} element are given as

$$\left.\begin{array}{l}\text{At node 1: } x = x_1;\ y = y_1;\ z = z_1, \text{ then } \omega = \omega_1 \\ \vdots \\ \text{At node 4: } x = x_4;\ y = y_4;\ z = z_4, \text{ then } \omega = \omega_4\end{array}\right\} \quad (1.279)$$

Substituting Eq.(1.279) into Eq.(1.277), one obtains:

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{Bmatrix} = \begin{bmatrix}1 & x_1 & y_1 & z_1\\ 1 & x_2 & y_2 & z_2\\ 1 & x_3 & y_3 & z_3\\ 1 & x_4 & y_4 & z_4\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\\ a_3\\ a_4\end{Bmatrix} \quad (1.280)$$

In a more compacted notations, Eq.(1.280) can be expressed as

$$\vec{\omega} = [A]_{4x4}\{a\}_{4x1} \quad (1.281)$$

From Eq.(1.281), one gets:

$$\{a\} = [A]^{-1}\vec{\omega} \quad (1.282)$$

Substituting Eq.(1.282) into Eq.(1.277), one obtains:

$$\omega(x, y, z) = [1, x, y, z]_{1x4} * [A]_{4x4}^{-1} * \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{Bmatrix} \quad (1.283)$$

or

$$\omega(x, y, z) = [N(x, y, z)]_{1x4} * \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{Bmatrix} \quad (1.284)$$

where the shape functions can be identified as

$$[N(x, y, z)]_{1x4} \equiv [1, x, y, z] * [A]^{-1} \quad (1.285)$$

Let $W=N_i(x, y, z)$, for i=1,2,3,4 (tetrahedral) (1.286)

and substituting Eq.(1.284) into Eq.(1.274), one obtains the following (finite) element equations:

$$0 = \int_{\Omega^e} \left[c_1 \frac{\partial(N_j\omega_j)}{\partial x} \frac{\partial N_i}{\partial x} + c_2 \frac{\partial(N_j\omega_j)}{\partial y} \frac{\partial N_i}{\partial y} + c_3 \frac{\partial(N_j\omega_j)}{\partial z} \frac{\partial N_i}{\partial z} \right] d\Omega - \int_{\Omega^e} N_i f d\Omega - \oint_{\Gamma^e} N_i q_n d\Gamma \tag{1.287}$$

or

$$\int_{\Omega^e} \left[c_1 \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + c_2 \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + c_3 \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right] d\Omega * \{\omega_j\} = \int_{\Omega^e} N_i f d\Omega + \oint_{\Gamma^e} N_i q_n d\Gamma \tag{1.288}$$

or

$$[k_{ij}^{(e)}]_{4x4} * \{\omega_j^{(e)}\}_{4x1} = \{F_i^{(e)}\} \tag{1.289}$$

where:

$$[k_{ij}^{(e)}] \equiv \int_{\Omega^e} \left[c_1 \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + c_2 \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + c_3 \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right] d\Omega$$

$$\{F_i^{(e)}\}_{4x1} = \int_{\Omega^e} N_i f d\Omega + \oint_{\Gamma^e} N_i q_n d\Gamma \tag{1.290}$$

The first term on the right side of Eq.(1.290) represents the equivalent joint loads due to the distributed "body" force "f", while the second term represents the equivalent joint loads due to the distributed "boundary" force "q_n".

1.14 Finite Element Weak Formulations For General 2-D Field Equations

The two-dimensional time-dependent field equation can be assumed in the following form:

$$c_1 \frac{\partial^2 u}{\partial x^2} + c_2 \frac{\partial^2 u}{\partial y^2} + c_3 \frac{\partial^2 u}{\partial x \partial y} + c_4 \frac{\partial u}{\partial x} + c_5 \frac{\partial u}{\partial y} + c_6 u^2 + c_7 u + c_8 + c_{11} \text{ctg}(u) \frac{\partial u}{\partial x} + c_{12} \text{ctg}(x) \frac{\partial u}{\partial x} = c_9 \frac{\partial^2 u}{\partial t^2} + c_{10} \frac{\partial u}{\partial t} \tag{1.291}$$

where $c_i, i = 1-12$ are constants; $u = u(x, y, t)$

It should be noted that the terms associated with constants c_{11} and c_{12} are included for handling other special applications [1.14].

The weighted residual equation can be established by the familiar procedure

$$\iint_{\Omega^e} w(c_1 \frac{\partial^2 u}{\partial x^2} + c_2 \frac{\partial^2 u}{\partial y^2} + c_3 \frac{\partial^2 u}{\partial x \partial y} + c_4 \frac{\partial u}{\partial x} + c_5 \frac{\partial u}{\partial y} + c_6 u^2 + c_7 u + c_8 - c_9 \frac{\partial^2 u}{\partial t^2} - c_{10} \frac{\partial u}{\partial t} + c_{11} \text{ctg}(u) \frac{\partial u}{\partial x} + c_{12} \text{ctg}(x) \frac{\partial u}{\partial x}) dxdy = 0 \tag{1.292}$$

where w ≡ Weighting functions.

The following relationships can be established through integration by parts:

$$\begin{aligned} & c_1 \iint_{\Omega^e} w \frac{\partial^2 u}{\partial x^2} dxdy \\ & = c_1 \iint_{\Omega^e} w \frac{\partial}{\partial x} (\frac{\partial u}{\partial x}) dxdy \\ & = c_1 \iint_{\Omega^e} [\frac{\partial}{\partial x}\left(w \frac{\partial u}{\partial x} \right) - \frac{\partial w}{\partial x} \frac{\partial u}{\partial x}] dxdy \\ & = c_1 \oint_{\Gamma^e} w \frac{\partial u}{\partial x} n_x ds - c_1 \iint_{\Omega^e} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dxdy; \end{aligned} \tag{1.293}$$

$$\begin{aligned} & c_2 \iint_{\Omega^e} w \frac{\partial^2 u}{\partial y^2} dxdy \\ & = c_2 \iint_{\Omega^e} w \frac{\partial}{\partial y} (\frac{\partial u}{\partial y}) dxdy \\ & = c_2 \iint_{\Omega^e} [\frac{\partial}{\partial y}\left(w \frac{\partial u}{\partial y} \right) - \frac{\partial w}{\partial y} \frac{\partial u}{\partial y}] dxdy \\ & = c_2 \oint_{\Gamma^e} w \frac{\partial u}{\partial y} n_y ds - c_2 \iint_{\Omega^e} \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} dxdy; \end{aligned} \tag{1.294}$$

$$
\begin{aligned}
& c_3 \iint_{\Omega^e} w \frac{\partial^2 u}{\partial x \partial y} dxdy \\
& = \frac{c_3}{2} \iint_{\Omega^e} w \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) dxdy + \frac{c_3}{2} \iint_{\Omega^e} w \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) dxdy \\
& = \frac{c_3}{2} \iint_{\Omega^e} \left[\frac{\partial}{\partial x}\left(w \frac{\partial u}{\partial y}\right) - \frac{\partial w}{\partial x}\frac{\partial u}{\partial y}\right] dxdy + \frac{c_3}{2} \iint_{\Omega^e} \left[\frac{\partial}{\partial y}\left(w \frac{\partial u}{\partial x}\right) - \frac{\partial w}{\partial y}\frac{\partial u}{\partial x}\right] dxdy \\
& = \frac{c_3}{2} \oint_{\Gamma^e} w \frac{\partial u}{\partial y} n_x ds + \frac{c_3}{2} \oint_{\Gamma^e} w \frac{\partial u}{\partial x} n_y ds - \frac{c_3}{2} \iint_{\Omega^e} \frac{\partial w}{\partial x}\frac{\partial u}{\partial y} dxdy - \frac{c_3}{2} \iint_{\Omega^e} \frac{\partial w}{\partial y}\frac{\partial u}{\partial x} dxdy;
\end{aligned}
\tag{1.295}
$$

Substituting Eqs.(1.293-1.295) into Eq.(1.292), one gets:

$$
\begin{aligned}
& \iint_{\Omega^e} \Big(-c_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} - c_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} - \frac{c_3}{2}\frac{\partial w}{\partial x}\frac{\partial u}{\partial y} - \frac{c_3}{2}\frac{\partial w}{\partial y}\frac{\partial u}{\partial x} - c_4 w \frac{\partial u}{\partial x} - c_5 w \frac{\partial u}{\partial y} \\
& \quad + c_6 w u^2 + c_7 w u + c_8 w - c_9 w \frac{\partial^2 u}{\partial t^2} - c_{10} w \frac{\partial u}{\partial t} \\
& \quad + c_{11} w\,\mathrm{ctg}(u) \frac{\partial u}{\partial x} + c_{12} w\,\mathrm{ctg}(x) \frac{\partial u}{\partial x}\Big) dxdy \\
& \qquad + \oint_{\Gamma^e} w \left[n_x \left(c_1 \frac{\partial u}{\partial x} + \frac{c_3}{2}\frac{\partial u}{\partial y} \right) + n_y \left(c_2 \frac{\partial u}{\partial y} + \frac{c_3}{2}\frac{\partial u}{\partial x} \right) \right] ds = 0
\end{aligned}
\tag{1.296}
$$

Let

$$
n_x \left(c_1 \frac{\partial u}{\partial x} + \frac{c_3}{2}\frac{\partial u}{\partial y} \right) + n_y \left(c_2 \frac{\partial u}{\partial y} + \frac{c_3}{2}\frac{\partial u}{\partial x} \right) \equiv q_n \tag{1.297}
$$

Then Eq.(1.296) becomes:

$$\iint_{\Omega^e}(-c_1\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}-c_2\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}-\frac{c_3}{2}\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}-\frac{c_3}{2}\frac{\partial w}{\partial y}\frac{\partial u}{\partial x}-c_4 w\frac{\partial u}{\partial x}$$

$$-c_5 w\frac{\partial u}{\partial y}+c_6 wu^2+c_7 wu+c_8 w-c_9 w\frac{\partial^2 u}{\partial t^2}-c_{10} w\frac{\partial u}{\partial t}$$

$$+c_{11} wctg(u)\frac{\partial u}{\partial x}+c_{12} wctg(x)\frac{\partial u}{\partial x})dxdy+\oint_{\Gamma^e} wq_n ds=0 \tag{1.298}$$

The dependent variable field u(x,y,t) is assumed to be in the following form:

$$u(x,y,t)\approx\sum_{j=1}^{n}u_j^e(t)\psi_j^e(x,y)\text{, where } n\equiv \text{ the dof per element} \tag{1.299}$$

Let the weighting function $w=\psi_i^e(x,y)$,(see Eq.1.292) , then Eq.(1.298) becomes

$$\iint_{\Omega^e}\left(\begin{array}{l}-c_1\frac{\partial\psi_i}{\partial x}(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial x})-c_2\frac{\partial\psi_i}{\partial y}(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial y})\\ -\frac{c_3}{2}\frac{\partial\psi_i}{\partial x}(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial y})-\frac{c_3}{2}\frac{\partial\psi_i}{\partial y}(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial x})\\ -c_4\psi_i(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial x})-c_5\psi_i(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial y})+c_6\psi_i(\sum_{j=1}^{n}u_j\psi_j)u\\ +c_7\psi_i(\sum_{j=1}^{n}u_j\psi_j)+c_8\psi_i+c_{11}\psi_i ctg(u)(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial x})\\ +c_{12}\psi_i ctg(x)(\sum_{j=1}^{n}u_j\frac{\partial\psi_j}{\partial x})-c_9\psi_i(\sum_{j=1}^{n}\frac{d^2u}{dt^2}\psi_j)\\ -c_{10}\psi_i(\sum_{j=1}^{n}\frac{du}{dt}\psi_j)\end{array}\right)dxdy$$

$$+\oint_{\Gamma^e} wq_n ds=0 \tag{1.300}$$

Eq.(1.300) can also be expressed as:

$$\sum_{j=1}^{n}\iint_{\Omega^e}\left[\begin{array}{l}(-c_1\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}-c_2\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}-\frac{c_3}{2}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y}\\ -\frac{c_3}{2}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x}-c_4\psi_i\frac{\partial\psi_j}{\partial x}-c_5\psi_i\frac{\partial\psi_j}{\partial y}+c_6u\psi_i\psi_j\\ +c_7\psi_i\psi_j+c_{11}\psi_i\text{ctgu}\frac{\partial\psi_j}{\partial x}+c_{12}\psi_i\text{ctgx}\frac{\partial\psi_j}{\partial x})u_j\\ -c_9\psi_i\psi_j\frac{d^2u_j}{dt^2}-c_{10}\psi_i\psi_j\frac{du_j}{dt})\end{array}\right]dxdy$$

$$+\iint_{\Omega^e}c_8\psi_i dxdy+\oint_{\Gamma^e}wq_n ds=0 \qquad (1.301)$$

In matrix form, Eq.(1.301) becomes:

$$[K^e]\{u^e\}+[C^e]\{\dot{u}^e\}+[M^e]\{\ddot{u}^e\}=\{f^e\}+\{Q^e\} \qquad (1.302)$$

where

$$K_{ij}^e=\iint_{\Omega^e}[(-c_1\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}-c_2\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}-\frac{c_3}{2}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial y}-\frac{c_3}{2}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial x}-c_4\psi_i\frac{\partial\psi_j}{\partial x}$$
$$-c_5\psi_i\frac{\partial\psi_j}{\partial x}+c_6u\psi_i\psi_j+c_7\psi_i\psi_j+c_{11}\text{ctg}(u)\psi_i\frac{\partial\psi_j}{\partial x}+c_{12}\text{ctg}(x)\psi_i\frac{\partial\psi_j}{\partial x}]dxdy \qquad (1.303)$$

$$C_{ij}^e=-\iint_{\Omega^e}c_{10}\psi_i\psi_j dxdy \qquad (1.304)$$

$$M_{ij}^e=-\iint_{\Omega^e}c_9\psi_i\psi_j dxdy \qquad (1.305)$$

$$f_i^e=-\iint_{\Omega^e}c_8\psi_i dxdy \qquad (1.306)$$

$$Q_i^e=-\oint_{\Gamma^e}\psi_i q_n ds \qquad (1.307)$$