Example 2.4.2 (Euler-Bernoulli Beam)

Consider the problem of finding the function $w(x)$ that satisfies the differential equation

$$\frac{d^2}{dx^2}\left[b(x)\frac{d^2w}{dx^2}\right]+c_f w-q(x)=0 \quad \text{for} \quad 0<x<L \tag{2.4.25}$$

This equation arises in the study of the elastic bending of beams (under the Euler-Bernoulli hypothesis that plane sections perpendicular to the axis of the beam before deformation remain plane after deformation), where w denotes the transverse deflection of the beam, L is the length of the beam, $b(x)>0$ is the flexural rigidity of the beam (i.e., the product of modulus of elasticity E and second moment of inertia I), c_f is the foundation modulus, and $q(x)$ is the transverse distributed load, as shown in Fig. 2.4.1. At the moment, we do not have to consider any specific boundary conditions of the problem.

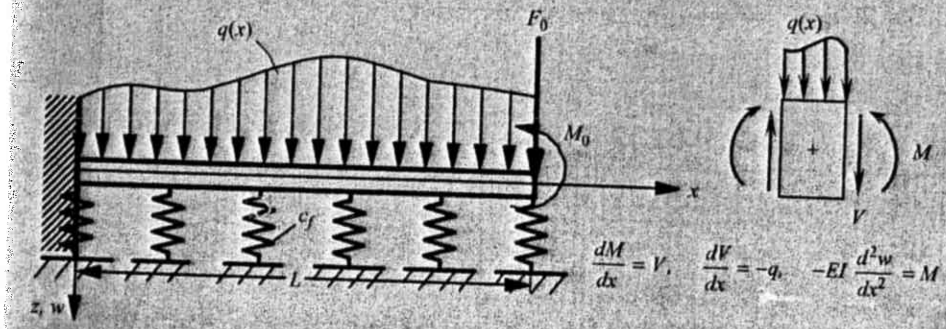


Figure 2.4.1 Cantilever beam with a set of loads.

page #1

Euler Beam Example # 2.4.2 (on pp. 67) of JN Reddy's 3rd Edition book

$$\underbrace{\frac{d^2}{dx^2}\left[b(x)\frac{d^2w}{dx^2}\right] + c_f w - q(x)}_{R(x)} = 0 \quad \text{for } 0 \le x \le L$$

<u>Notes</u>:

(a) $2m = 4^{th}$ derivatives $\Longrightarrow \boxed{m-1=1}$

any b.c. that involves with derivatives upto & include order 1 are <u>Geo. b.c.</u>, such as

$w(@x=0) = \cdots$

$w'(@x=0) = \cdots$

Natural b.c., such as

$w''(@x=L) = \cdots$ → Moment

$w'''(@x=L) = \cdots$ → shear force

(b) At each node, we have 2 unknown dof, such as $w(@\text{ node } i)$
$w'(@\text{ node } i)$

(c) <u>Weak Formulation</u>:

step 1 : $\int_0^L v\,[R(x)]\,dx = 0$

step 2 : Integrate by part TWICE $\Longrightarrow$ combined 2 Boundary Terms (B.T.)

step 3 : Apply the given Natural b.c. into B.T.

to get $B(v,w) = \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f v w\right)dx$ → SYM. operator! $\Longrightarrow$ get functional $I(w) = \text{----}$ (see JNReddy's page 69)

$\ell(v) = \int_0^L v q\,dx + \left[v(@L)F_0 - \left.\frac{dv}{dx}\right|_{@L} * M_0\right]$

#2

Since the equation contains a fourth-order derivative, we should integrate it twice by parts to distribute the derivatives equally between the dependent variable w and the weight function v. In this case, v must be twice differentiable and satisfy the homogeneous form of an EBC. Multiplying (2.4.25) by v, and integrating the first term by parts twice with respect to x, we obtain

Step 1: $$0 = \int_0^L v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) + c_f w - q\right] dx \tag{2.4.26}$$

Step 2: $$\begin{aligned} 0 &= \int_0^L \left[\left(-\frac{dv}{dx}\right)\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) + c_f vw - vq\right] dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_0^L \\ &= \int_0^L \left(\frac{d^2v}{dx^2}b\frac{d^2w}{dx^2} + c_f vw - vq\right) dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}b\frac{d^2w}{dx^2}\right]_0^L \end{aligned} \tag{2.4.27}$$

From the last line, it follows that the specification of w and dw/dx constitutes the essential (geometric or static) boundary conditions, and the specification of

$$\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) \text{ (shear force)} \quad \text{and} \quad b\left(\frac{d^2w}{dx^2}\right) \text{ (bending moment)} \tag{2.4.28a}$$

constitutes the natural boundary conditions for the Euler-Bernoulli beam theory.

Now consider a specific beam problem. Let us consider a beam fixed at the left end and subjected to transverse force and bending moment at $x = L$, as shown in Fig. 2.4.1:

$$w(0) = 0, \quad \left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad \left(b\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = -M_0, \quad \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x=L} = -F_0 \tag{2.4.28b}$$

where M_0 is the bending moment and F_0 is the transverse load. Since w and dw/dx (both are primary variables) are specified at $x = 0$, we require the weight function v and its derivative dv/dx to be zero there

$$v(0) = \left(\frac{dv}{dx}\right)\bigg|_{x=0} = 0$$

The remaining two boundary conditions in (2.4.28b) are natural boundary conditions, which place no restrictions on v and its derivatives. Thus, Eq. (2.4.27) becomes

Step 3 $$0 = \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right) dx - v(L)F_0 + \left(\frac{dv}{dx}\right)\bigg|_{x=L} M_0 \tag{2.4.29}$$

Variational Problem and Quadratic Functional: Equation (2.4.30a) can be written in the form

$$B(v, w) = l(v) \tag{2.4.30a}$$

where

$$\begin{aligned} B(v, w) &= \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw\right) dx \\ l(v) &= \int_0^L vq\, dx + v(L)F_0 - \left(\frac{dv}{dx}\right)\bigg|_{x=L} M_0 \end{aligned} \tag{2.4.30b}$$

#3

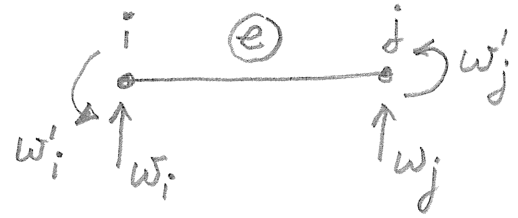
The functional, known as the total potential energy of the beam, is obtained using (2.4.19):

$$I(w) = \int_0^L \left[\frac{b}{2} \left(\frac{d^2 w}{dx^2} \right)^2 + \frac{c_f}{2} w^2 - w q \right] dx - w(L) F_0 + \left(\frac{dw}{dx} \right) \bigg|_{x=L} M_0 \tag{2.4.31}$$

Note that for the fourth-order equation, the essential boundary conditions involve not only the dependent variable but also its first derivative. As pointed out earlier, at any boundary point, only one of the two boundary conditions (essential or natural) can be specified. For example, if the transverse deflection is specified at a boundary point, then one cannot specify the shear force V at the same point, and vice versa. Similar comments apply to the slope dw/dx and the bending moment M. Note that in the present case, w and dw/dx are the primary variables, and V and M are the secondary variables.

#4

$\boxed{\text{Step 4}}$



Let $w(x) = \overset{?}{a_1} + \overset{?}{a_2} x + \overset{?}{a_3} x^2 + \overset{?}{a_4} x^3 = [1, x, x^2, x^3] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$ (Why ??)

Hence $w'(x) = a_2 + 2a_3 x + 3a_4 x^2$

<u>Apply 4 Geo. b.c. for a typical element</u>

$$
\begin{cases}
@\ x = x_i \Rightarrow w(x) = w_i = a_1 + a_2 x_i + a_3 x_i^2 + a_4 x_i^3 \\
@\ x = x_i \Rightarrow w'(x) = w'_i = a_2 + 2a_3 x_i + 3a_4 x_i^2
\end{cases}
$$

$$@\ x = x_j \Rightarrow w(x) = w_j = a_1 + a_2 x_j + \cdots$$

$$@\ x = x_j \Rightarrow w'(x) = w'_j = a_2 + 2a_3 x_j + \cdots$$

$$
\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}
\overset{?}{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}}
=
\begin{Bmatrix} w_i \\ w'_i \\ w_j \\ w'_j \end{Bmatrix}
$$

$$[A]\,\vec{a} = \vec{u_i} \Rightarrow \vec{a} = [4 \times 4]^{-1} \vec{u_i}$$

Thus $w(x) = \underbrace{[1, x, x^2, x^3]\,[A]^{-1}}\,\vec{u_i} = \sum_{i=1}^{N=4\text{ dof per beam element}} \phi_i(x)\, u_i$

$= [\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)] * \begin{Bmatrix} u_1 = w_i \\ u_2 = w'_i \\ u_3 = w_j \\ u_4 = w'_j \end{Bmatrix}$

Shape function

$\boxed{\text{Step 5}}$

Recalled $B(v,w) = \displaystyle\int_0^L \left(b \, \frac{d^2 \overset{\phi_j}{v}}{dx^2} \, \frac{d^2 \overset{\sum u_i \phi_i}{w}}{dx^2} + c_f \, \overset{\phi_j}{v} \, \overset{u_i \phi_i}{w} \right) dx$

$$B(v,w) = \underbrace{\int_0^L \left(b \, \phi_j'' * \phi_i'' \cancel{u_i} + c_f \, \phi_j \phi_i \cancel{u_i} \right) dx}_{\textstyle [K_{ij}]_{4\times4}} * \vec{u}_i$$

$$B(v,w) = \underset{4\times4}{[K_{ij}]} * \left\{ \vec{u}_i = \begin{matrix} w_i \\ w_i' \\ w_j \\ w_j' \end{matrix} \right\}$$

$$l(v) = \int_0^L \underset{\phi_i(x)}{v} q \, dx + \left[\underset{\phi_i(@x=L)}{\cancel{v(@L)}} F_0 - \underset{\phi_i'\big|_{@x=L}}{\left.\frac{dv}{dx}\right|_{@L}} M_0 \right] \equiv \vec{f}_i$$

since $B(v,w) = l(v)$

Hence $[K_{ij}^{(e)}] \, \vec{u}_j^{(e)} = \vec{f}_i^{(e)}$

<u>Step 6</u>:

$$\left.\begin{aligned} &\text{system } [K] = \sum_e K_{ij}^{(e)} \\ &\text{system } \vec{F} = \sum_e \vec{f}_i^{(e)} \end{aligned}\right\} \Rightarrow \underset{\text{singular}}{[K]} \, \vec{U} = \vec{F}$$

<u>Step 7</u>: Apply Geo. b.c. @ system Level

$$\underset{\text{non-singular}}{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 1 & & \\ & & 0 & & & \end{bmatrix}} \left\{ \vec{U} \right\} = \underset{\text{All Knowns}}{\left\{ \vec{F}^* \right\}}$$

#6

Example 2.4.4 (Poisson's Equation in Two Dimensions)

Consider the problem of determining the solution $u(x, y)$ to the partial differential equation,

$$-\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)+a_0 u=f \quad \text{in } \Omega \tag{2.4.41}$$

in a closed two-dimensional domain Ω with boundary Γ, as shown in Fig. 2.4.2. Here a_0, a_1, a_2, and f are known functions of position (x, y) in Ω. The function u is required to satisfy, in addition to the differential equation (2.4.41), certain boundary conditions on the boundary Γ of Ω. The weak formulation to be discussed next will reveal the precise form of the essential and natural boundary conditions of the equation.

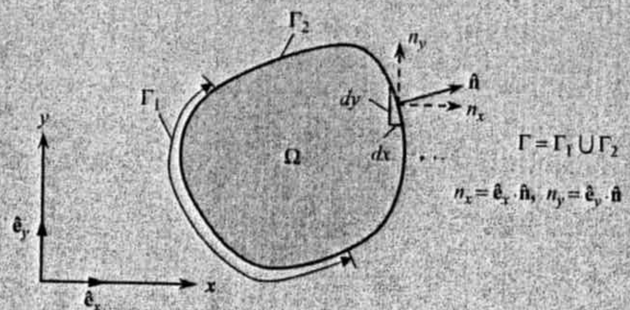


Figure 2.4.2 A two-dimensional domain Ω with boundary Γ.

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How to handle 2-D ODE problems ??

See J. N. Reddy's Example 2.4.4 (pp. 71)

$2m = 2 \Rightarrow \boxed{m-1=0}$

Weak Formulation

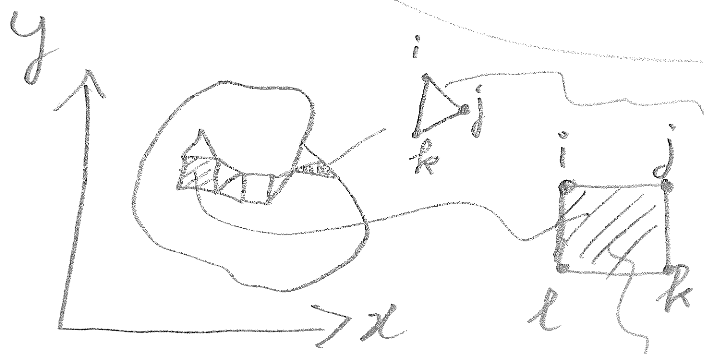
Step 1
Step 2
Step 3

Hence: $B(w,u) = \iint_{\Omega} \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u \right) dx\,dy$

SYM.

Get functional $I(u) = \frac{1}{2}\iint_{\Omega} [\cdots\cdots]\,dx\,dy - \iint_{\Omega} uf\,dx\,dy - \int_{\Gamma_2} \hat{q} u$

See JNR, page 72



$u(x,y) = a_1 + a_2 x + a_3 y$

(WHY ?)

Then How to get 3 shape functions?

$u(x,y) = b_1 + b_2 x + b_3 y + \cancel{b_4} x^2 + b_5 xy + \cancel{b_6 y^2}$

(WHY ??)

(How to get 4 shape functions?)

The three step procedure applied to Eq. (2.4.41) results in the following equations:

Step 1: $$0 = \int_\Omega w \left[-\frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2 \frac{\partial u}{\partial y}\right) + a_0 u - f \right] dxdy \qquad (2.4.42)$$

Step 2: $$0 = \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u - w f \right) dxdy - \oint_\Gamma w \left(a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y \right) ds \qquad (2.4.43)$$

where we used integration by parts [or the gradient and divergence theorems, Eqs. (2.2.27*b*) and (2.2.28*b*)] to transfer differentiation from u to w so that both u and w have the same order derivatives in Ω. The boundary term shows that u is the primary variable while

$$a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y$$

is the secondary variable.

Step 3: The last step in the procedure is to impose the specified boundary conditions. Suppose that u is specified on portion Γ_1 and the natural boundary condition is specified on the remaining portion Γ_2 of the boundary as (see Fig. 2.4.2):

$$u = \hat{u} \ \text{ on } \Gamma_1, \quad a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y = \hat{g} \ \text{ on } \Gamma_2 \qquad (2.4.44)$$

Then w is arbitrary on Γ_2 and equal to zero on Γ_1. Consequently, Eq. (2.4.43) simplifies to

$$0 = \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u - w f \right) dxdy - \int_{\Gamma_2} w \hat{g}\, ds \qquad (2.4.45)$$

Variational Problem and Quadratic Functional: The weak statement (2.4.45) can be expressed as $B(w,u) = \ell(w)$, where the bilinear form and linear form are

$$B(w,u) = \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u \right) dxdy, \qquad (2.4.46a)$$

$$\ell(w) = \int_\Omega w f \, dxdy + \int_{\Gamma_2} w \hat{g}\, ds, \qquad (2.4.46b)$$

or

The associated quadratic functional is $I(u) = \frac{1}{2} B(u,u) - \ell(u)$

$$I(u) = \frac{1}{2}\int_\Omega \left[a_1 \left(\frac{\partial u}{\partial x}\right)^2 + a_2 \left(\frac{\partial u}{\partial y}\right)^2 + a_0 u^2 \right] dxdy - \int_\Omega u f \, dxdy - \int_{\Gamma_2} \hat{g} u \, ds. \qquad (2.4.46c)$$

In the case of transverse deflections of a membrane, $I(u)$ represents the total potential energy.

#9

As a specific example of the Poisson equation, consider steady heat conduction in a two-dimensional domain Ω, enclosed by lines AB, BC, CD, DE, EF, FG, GH, and HA, as indicated

Example 2.4.3 (Timoshenko Beam)

Consider the following pair of coupled differential equations governing bending of the Timoshenko beam:

$$-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]+c_f w = q \tag{2.4.32a}$$

$$-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right)+S\left(\frac{dw}{dx}+\phi_x\right)=0 \tag{2.4.32b}$$

where S is the shear stiffness ($S = K_s GA$; K_s is the shear correction coefficient, G is the shear modulus, and A is the area of the cross section), $D = EI$ is the bending stiffness, w is the transverse deflection, ϕ_x is the rotation, k is the foundation modulus, and q is the distributed transverse load. We shall develop the weak form of the above equations using the three-step procedure.

Steps 1 and 2: Multiply the first equation with weight function v_1 and the second one with weight function v_2 and integrate over the length of the beam:

Step 1*a*: $$0=\int_0^L v_1\left\{-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]+c_f w-q\right\}dx$$

Step 2*a*: $$\begin{aligned}0&=\int_0^L\left[\frac{dv_1}{dx}S\left(\frac{dw}{dx}+\phi_x\right)+c_f v_1 w-v_1 q\right]dx-\left[v_1 S\left(\frac{dw}{dx}+\phi_x\right)\right]_0^L\\&=\int_0^L\left[\frac{dv_1}{dx}S\left(\frac{dw}{dx}+\phi_x\right)+c_f v_1 w-v_1 q\right]dx-v_1(L)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=L}\\&\quad+v_1(0)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=0}\end{aligned} \tag{2.4.33a}$$

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How to handle system of 2 (or more) ODE ??

See JN Reddy's Timoshenko example, on pp. 69 #2.4.3

Notes: for 1st primary variable $w_{(x)} \Rightarrow 2m = 2^{th}$ derivatives

Try/select $w_{(x)} = a_1 + a_2 x = [1, x] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \Rightarrow w(x) = [\phi_1(x), \phi_2(x)] \{ \}$

$\Leftarrow \boxed{m-1=0}$

for 2nd primary variable $\phi_x \Rightarrow 2m = 2^{th}$ derivatives

$\Downarrow$

$\boxed{m-1=0}$

Try/select $\phi_x(x) = a_3 + a_4 x = [1, x] \begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix}$

$= etc. \ldots$

$= [\phi_3(x), \phi_4(x)] \begin{Bmatrix} \phi_{x_i} \\ \phi_{x_j} \end{Bmatrix}$

$$\textit{Step 1b}: \quad 0 = \int_0^L v_2 \left[-\frac{d}{dx}\left(D\frac{d\phi_x}{dx} \right) + S\left(\frac{dw}{dx} + \phi_x \right) \right] dx$$

$$\textit{Step 2b}: \quad 0 = \int_0^L \left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + v_2\, S\left(\frac{dw}{dx} + \phi_x \right) \right] dx - \left[v_2\, D\frac{d\phi_x}{dx} \right]_0^L$$

$$= \int_0^L \left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + v_2\, S\left(\frac{dw}{dx} + \phi_x \right) \right] dx$$

$$- v_2(L) \left[D\frac{d\phi_x}{dx} \right]_{x=L} + v_2(0) \left[D\frac{d\phi_x}{dx} \right]_{x=0} \qquad (2.4.33b)$$

Note that integration-by-parts was used such that the expression $w_{,x} + \phi_x$ is preserved, as it enters the boundary term representing the shear force. Such considerations can only be used by knowing the mechanics of the problem at hand. Also, note that the pair of weight functions (v_1, v_2) satisfy the homogeneous form of specified essential boundary conditions on the pair (w, ϕ_x) (with the correspondence $v_1 \sim w$ and $v_2 \sim \phi_x$).

Steps 3: An examination of the boundary terms shows that $w \sim v_1$ and $\phi_x \sim v_2$ are the primary variables, and the secondary variables are given by [cf. Eq. (2.4.28a)]

$$S\left(\frac{dw}{dx} + \phi_x \right) \quad \text{(shear force)} \qquad (2.4.34a)$$

$$D\frac{d\phi_x}{dx} \quad \text{(bending moment)} \qquad (2.4.34b)$$

To finalize the weak forms, we must take care of the boundary terms by considering a specific beam problem. Using the beam of Fig. 2.4.1, we see that

$$w(0) = 0, \quad \phi_x(0) = 0, \quad \left[S\left(\frac{dw}{dx} + \phi_x \right) \right]_{x=L} = F_0, \quad \left[D\frac{d\phi_x}{dx} \right]_{x=L} = M_0 \qquad (2.4.35)$$

and hence, $v_1(0) = 0$ and $v_2(0) = 0$. Consequently, the weak forms (2.4.33a) and (2.4.33b) become

$$0 = \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx} + \phi_x \right) + c_f v_1 w - v_1 q \right] dx - v_1(L)\, F_0 \qquad (2.4.36a)$$

$$0 = \int_0^L \left[\frac{dv_2}{dx} D\frac{d\phi_x}{dx} + v_2\, S\left(\frac{dw}{dx} + \phi_x \right) \right] dx - v_2(L)\, M_0 \qquad (2.4.36b)$$

Variational Problem and Quadratic Functional: To write the variational problem of finding (w, ϕ_x) such that

weight function, unknown, dependent function, weight function

$$B((v_1, v_2), (w, \phi_x)) = l((v_1, v_2)) \qquad (2.4.37)$$

holds for all (v_1, v_2), we must combine the two weak forms into a single expression

$$0 = \int_0^L \left[\left(\frac{dv_1}{dx} + v_2 \right) S\left(\frac{dw}{dx} + \phi_x \right) + \frac{dv_2}{dx} D\frac{d\phi_x}{dx} + c_f v_1 w - v_1 q \right] dx - v_1(L)\, F_0 - v_2(L)\, M_0$$

(2.4.38)

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Thus, the bilinear and linear forms of the problem are given by

swap ⇒ no change

$$B((v_1, v_2), (w, \phi_x)) = \int_0^L \left[S\left(\frac{dv_1}{dx} + v_2\right)\left(\frac{dw}{dx} + \phi_x\right) + D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + c_f v_1 w \right] dx \quad (2.4.39a)$$

$$l((v_1, v_2)) = \int_0^L v_1 q \, dx + v_1(L) F_0 + v_2(L) M_0 \quad (2.4.39b)$$

Clearly, $B((v_1, v_2), (w, \phi_x))$ is symmetric in its arguments (i.e. interchange of v_1 with w and v_2 with ϕ_x yields the same expression). Hence the functional is given by

$$I((w, \phi_x)) = \frac{1}{2}\int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + \frac{c_f}{2} w^2 \right] dx - \left(\int_0^L w q \, dx + w(L) F_0 + \phi_x(L) M_0 \right) \quad (2.4.40)$$

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