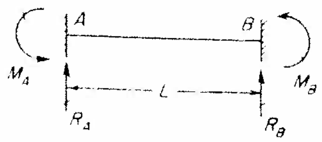
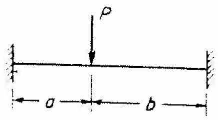
Fixed-End Actions Caused by Loads



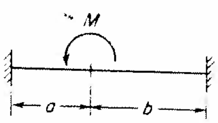
1



$$M_A = \frac{Pab^2}{L^2} \qquad M_B = -\frac{Pa^2b}{L^2} = \frac{PL}{8}$$

$$R_A = \frac{Pb^2}{L^3}(3a + b) \qquad R_B = \frac{Pa^2}{L^3}(a + 3b)$$

2

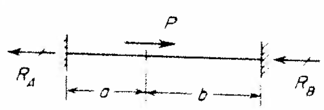


$$M_A = \frac{Mb}{L^2}(2a - b)$$

$$M_B = \frac{Ma}{L^2}(2b - a)$$

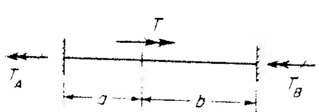
$$R_A = -R_B = \frac{6Mab}{L^3}$$

3



$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

4



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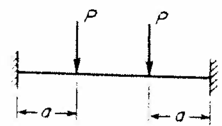
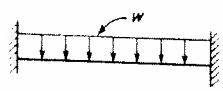


$$M_A = -M_B = \frac{Pa}{L}(L - a)$$

$$R_A = R_B = P$$

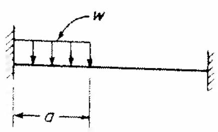
6



$$M_A = -M_B = \frac{wL^2}{12}$$

$$R_A = R_B = \frac{wL}{2}$$

7



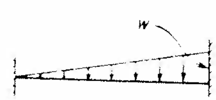
$$M_A = \frac{wa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = -\frac{wa^3}{12L^2}(4L - 3a)$$

$$R_A = \frac{wa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \frac{wa^3}{2L^3}(2L - a)$$

8



$$M_A = \frac{wL^2}{30} \qquad M_B = -\frac{wL^2}{20}$$

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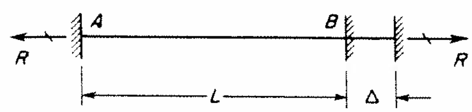
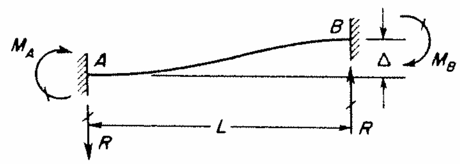
Fixed-End Actions Caused by End-Displacements

1



$$R = \frac{EA\Delta}{L}$$

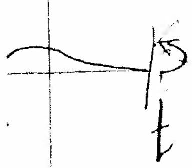
2

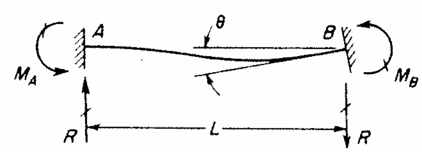


$$M_A = M_B = \frac{6EI\Delta}{L^2} \qquad R = \frac{12EI\Delta}{L^3}$$

3

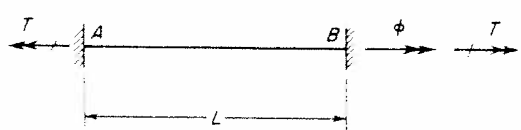
1 rad





$$M_A = \frac{2EI\theta}{L} \qquad M_B = \frac{4EI\theta}{L} \qquad R = \frac{6EI\theta}{L^2}$$

4



$$T = \frac{GJ\phi}{L}$$

G = shear modulus of elasticity

J = torsion constant

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Fixed End Moments (FEM) Due To Applied Loads

Using superposition principle, one obtains:

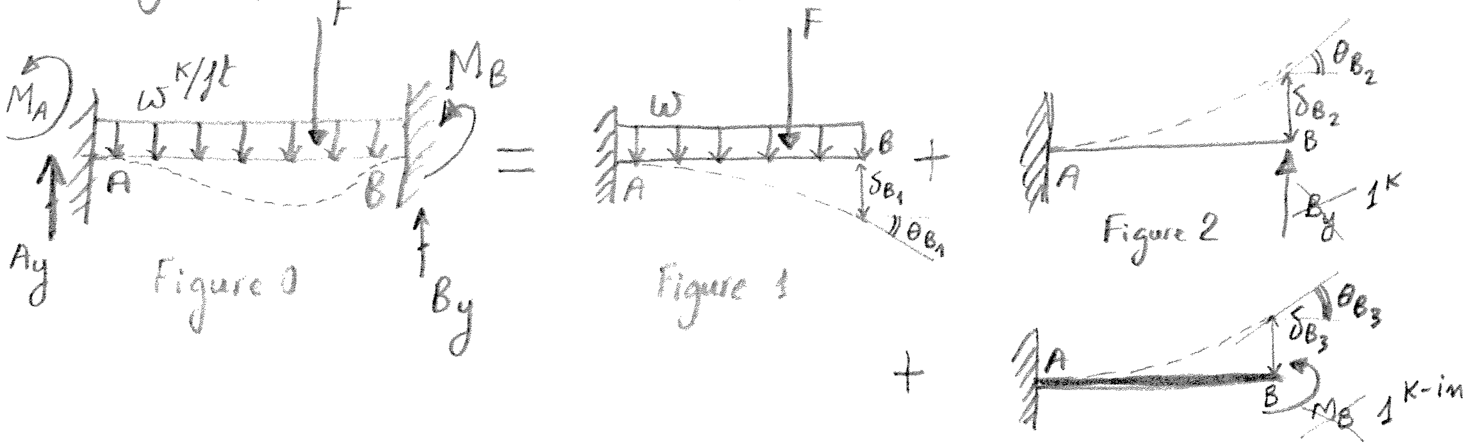


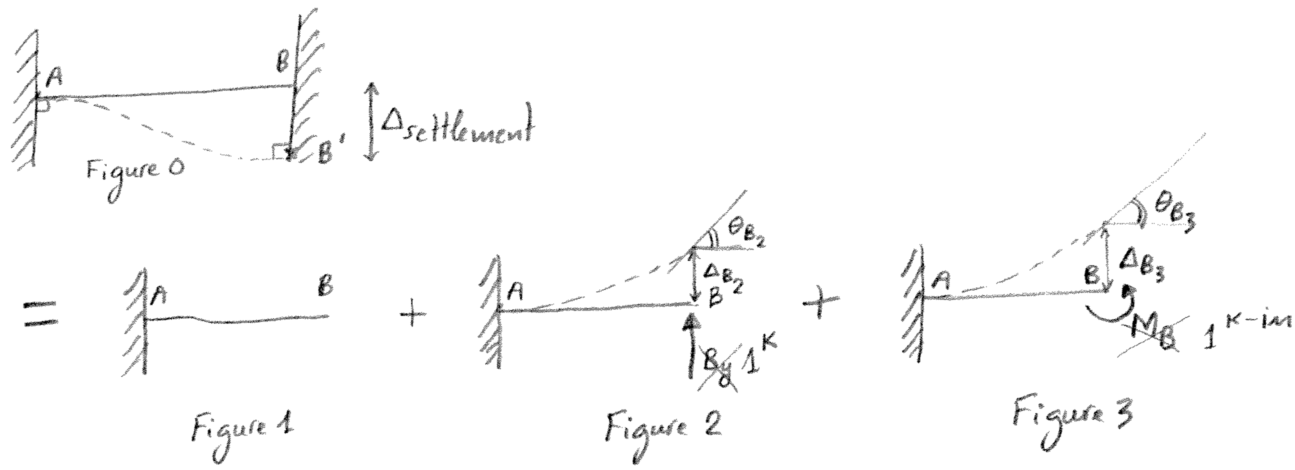
Figure 0 = Figure 1 + Figure 2 + Figure 3

$$\left(\delta_{B\,vertical}\right)_0 \equiv 0 = \left(\delta_{B\,vertical}\right)_1 + \left(\delta_{B\,vertical}\right)_2 * B_y + \left(\delta_{B\,vertical}\right)_3 * M_B$$

$$\left(\theta_B\right)_0 \equiv 0 = \left(\theta_B\right)_1 + \left(\theta_B\right)_2 * B_y + \left(\theta_B\right)_3 * M_B$$

From the above 2 "key" equations, one can solve for the unknown reactions B_y and M_B (at the fixed support B). Then, from statics equilibrium equations, the unknown support reactions A_y and M_A (at the fixed support A) can also be found.

FEM Due To (say, earthquake) Support Vertical Displacement



$$\Delta_{B_0} \equiv \Delta_{settlement} = \Delta_{B_2} * B_y + \Delta_{B_3} * M_B \qquad (1)$$

$$\theta_{B_0} \equiv 0 = \theta_{B_2} * B_y + \theta_{B_3} * M_B \qquad (2)$$

where (using ~~Moment Area Theorems~~, or Virtual Work Method)

$$\left.\begin{aligned} \Delta_{B_2} &= \frac{L^3}{3} \;;\quad \theta_{B_2} = \frac{L^2}{2} \\ \Delta_{B_3} &= \frac{L^2}{2} \;;\quad \theta_{B_3} = L \end{aligned}\right\} \qquad (3)$$

Using Eq. (3), then Eqs. (1-2) can be simultaneously solved for

$$\boxed{\begin{aligned} B_y &= \frac{+12\,\Delta_{settlement}}{L^3} \\ M_B &= \frac{-6\,\Delta_{settlement}}{L^2} \end{aligned}}$$

FEM Due to (say, earthquake) Support Rotation

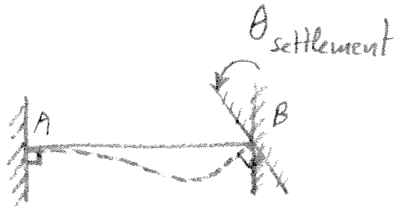


Figure 0

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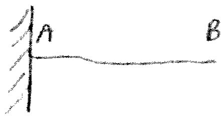

Figure 1

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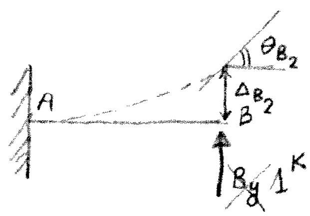

Figure 2

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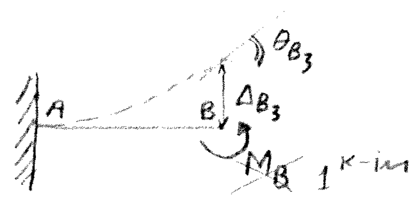

Figure 3

$$\Delta_{B_0} \equiv 0 = \Delta_{B_2} * B_y + \Delta_{B_3} * M_B \qquad (1)$$

$$\theta_{B_0} \equiv \theta_{settlement} = \theta_{B_2} * B_y + \theta_{B_3} * M_B \qquad (2)$$

Hence:

$$\boxed{\begin{aligned} M_B &= \frac{4\theta_{settlement}}{L} \\ B_y &= \frac{-6\theta_{settlement}}{L^2} \end{aligned}}$$

Fig. 3-1a. This beam has a fixed support at A and a roller support at B; and it is subjected to a uniform load of intensity w. The beam is kinematically indeterminate to the first degree (if axial deformations are neglected) because the only unknown joint displacement is the rotation θ_B at joint B. The first phase of the analysis is to determine this rotation. Then the various actions and displacements throughout the beam can be determined, as will be shown later.

In the flexibility method a statically determinate released structure is obtained by altering the actual structure in such a manner that the selected redundant actions are zero. The analogous operation in the stiffness method is to obtain a kinematically determinate structure by altering the actual structure in such a manner that all unknown displacements are zero. Since the unknown displacements are the translations and rotations of the joints, they can be made equal to zero by restraining the joints of the structure against displacements of any kind. The structure obtained by restraining all joints of the actual structure is called the *restrained structure*. For the beam in Fig. 3-1a the restrained structure is obtained by restraining joint B against rotation. Thus, the restrained structure is the fixed-end beam shown in Fig. 3-1b.

When the loads act on the restrained beam (see Fig. 3-1b), there will be a couple M_B developed at support B. This reactive couple is in the clockwise direction and is given by the expression

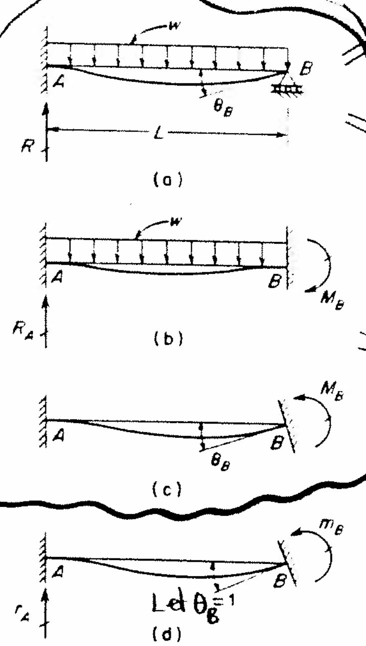

Fig. 3-1. Illustration of stiffness method.

$$(\Delta_{B\,vert})_{orig} = (\Delta_{B\,vert})_I + (\Delta_{B\,vert})_{II} * \overset{?}{B_y} = 0$$

or

$$(\theta_A)_{orig} = (\theta_B)_I + (\theta_B)_{II} * \overset{?}{M_{A\,left}} = 0$$

$$(M_B)_{orig} = (M_B)_{Fig(b)} + (m_B)_{Fig(d)} * \overset{?}{\theta_B}$$

zero; $-\frac{wL^2}{12}$; $\frac{4EI}{L}$

Actions due to load in the restraint structure

$$\vec{A}_D = \vec{A}_{DL} + [S]\vec{D}$$

$$\vec{A}_R = \vec{A}_{RL} + [A_{RD}]\vec{D}$$

$$\vec{A}_M = A_{ML} + [A_{MD}]\vec{D}$$

Actions applied @ joints correspond to unknown displ.

which can be found from tl
B (see Table B-1). Note tl
the rotation θ_B, which is
there is no couple at joint
to consider next that the r
opposite to the couple M,
Fig. 3-1c. When the acti
superimposed, they produ
ysis of the beam in Fig. 3
analyses shown in Figs. 3
tion produced by the cou
rotation in the actual bea

The relation between
of Fig. 3-1c is

in which EI is the flexur
from Case 3 of Table B-
M_B from Eqs. (3-1) and (

from which

Thus, the rotation at joi

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formulate the equation
to use a consistent sign
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The effect of a unit
where the restrained b
unit value of the rotati
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that rotation (while all
that m_B is a *stiffness c*
The value of the coupl

$$m_B = \frac{4EI}{L}$$

In formulating the equation of superposition the couples at joint B will be superimposed as follows. The couple in the restrained beam subjected to the load (Fig. 3-1b) will be added to the couple m_B (corresponding to a unit value of θ_B) multiplied by θ_B itself. The sum of these two terms must give the couple at joint B in the actual beam, which is zero in this example. All terms in the superposition equation will be expressed in the same sign convention, namely, that couples and rotations at joint B are positive when counterclockwise. According to this convention, the couple M_B in the beam of Fig. 3-1b is negative:

$$M_B = -\frac{wL^2}{12}$$

The equation for the superposition of moments at support B now becomes

$$M_B + m_B\theta_B = 0 \qquad (3\text{-}3)$$

or

$$-\frac{wL^2}{12} + \frac{4EI}{L}\theta_B = 0$$

Solving this equation yields

$$\theta_B = \frac{wL^3}{48EI}$$

which is the same result as before. The positive sign for the result means that the rotation is counterclockwise.

The most essential part of the preceding solution consists of writing the action superposition equation (3-3), which expresses the fact that the moment at B in the actual beam is zero. Included in this equation are the moment caused by the loads on the restrained structure and the moment caused by rotating the end B of the restrained structure. The latter term in the equation was expressed conveniently as the product of the moment caused by a unit value of the unknown displacement (stiffness coefficient) times the unknown displacement itself. The two effects are summed algebraically, using the same sign convention for all terms in the equation. When the equation is solved for the unknown displacement, the sign of the result will give the true direction of the displacement. The equation may be referred to either as an *equation of action superposition* or as an *equation of joint equilibrium*. The latter name is used because the equation may be considered to express the equilibrium of moments at joint B.

Having obtained the unknown rotation θ_B for the beam, it is now possible to calculate other quantities, such as member end-actions and reac-

tions. As an example, assume t
of the beam (Fig. 3-1a) is to b
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r_A in Fig. 3-1d, as shown in the

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The forces R_A and r_A can be re
Case 6, Table B-1, and Case 3

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The same concepts can be us
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must be found first.

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more organized approach to t
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stiffness method (see Fig. 3-2
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wise rotations are positive.
mined by solving equations
C, as described in the follov

The first step in the anal
the joints to prevent all joint
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end beams. The restrained
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Thus, only the loads P_1, P_2
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…ents at joint B.
…, for the beam, it is now pos-
…ember end-actions and reac-

tions. As an example, assume that the reactive force R acting at support A of the beam (Fig. 3-1a) is to be found. This force is the sum of the corresponding reactive force R_A at support A in Fig. 3-1b and θ_B times the force r_A in Fig. 3-1d, as shown in the following superposition equation:

$$R = R_A + \theta_B r_A$$

The forces R_A and r_A can be readily calculated for the restrained beam (see Case 6, Table B-1, and Case 3, Table B-4):

$$R_A = \frac{wL}{2} \qquad r_A = \frac{6EI}{L^2}$$

When these values, as well as the previously found value for θ_B, are substituted into the equation above, the result is

$$R = \frac{5wL}{8}$$

The same concepts can be used to calculate any other actions or displacements for the beam. However, in all cases the unknown joint displacements must be found first.

If a structure is kinematically indeterminate to more than one degree, a more organized approach to the solution, as well as a generalized notation, must be introduced. For this purpose, the same two-span beam used previously as an example in the flexibility method will be analyzed now by the stiffness method (see Fig. 3-2a). The beam has constant flexural rigidity EI and is subjected to the loads P_1, M, P_2, and P_3. Since rotations can occur at joints B and C, the structure is kinematically indeterminate to the second degree when axial deformations are neglected. Let the unknown rotations at these joints be D_1 and D_2, respectively, and assume that counterclockwise rotations are positive. These unknown displacements may be determined by solving equations of superposition for the actions at joints B and C, as described in the following discussion.

The first step in the analysis consists of applying imaginary restraints at the joints to prevent all joint displacements. The restrained structure which is obtained by this means is shown in Fig. 3-2b and consists of two fixed-end beams. The restrained structure is assumed to be acted upon by all of the loads except those that correspond to the unknown displacements. Thus, only the loads P_1, P_2, and P_3 are shown in Fig. 3-2b. All loads that correspond to the unknown joint displacements, such as the couple M in this example, are taken into account later. The moments A_{DL1} and A_{DL2} (Fig. 3-2b) are the actions of the restraints (against the restrained structure) corresponding to D_1 and D_2, respectively, and caused by loads acting on the structure. For example, the restraint action A_{DL1} is the sum of the reactive moment at B due to the load P_1 acting on member AB and the reactive moment at B due to the load P_2 acting on member BC. These actions can

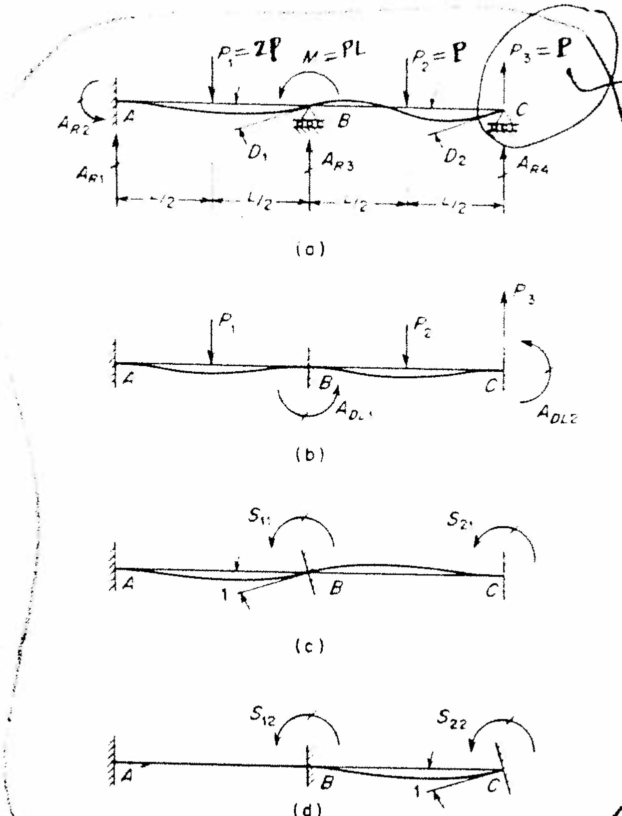

Fig. 3-2. Illustration of stiffness method.

No effect on Ft-displ. However, it will have effect on Reaction computation.

be found with the aid of formulas for fixed-end moments in beams (see Appendix B), as illustrated later.

In order to generate the stiffness coefficients at joints B and C, unit values of the unknown displacements D_1 and D_2 are induced separately in the restrained structure. A unit displacement corresponding to D_1 consists of a unit rotation of joint B, as shown in Fig. 3-2c. The displacement D_2 remains equal to zero in this beam. Thus, the actions corresponding to D_1 and D_2 are the stiffness coefficients S_{11} and S_{21}, respectively. These stiffnesses consist of the couples exerted by the restraints on the beam at joints B and C, respectively. The calculation of these actions is not difficult when formulas for fixed-end moments in beams are available. Their determination in this example will be described later. The condition that D_2 is equal to unity while D_1 is equal to zero is shown in Fig. 3-2d. In the figure the stiffness S_{12} is the action corresponding to D_1 while the stiffness S_{22} is the action corresponding to D_2. Note that in each case the stiffness coefficient is the action that the artificial restraint exerts upon the structure.

Two superposition equations expressing the conditions pertaining to the moments acting on the original structure (Fig. 3-2a) at joints B and C may

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equations express the fac
3-2a) are equal to the corr
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also be denoted $\mathbf{A}_{DD}$, s
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$$\mathbf{A}_D = \begin{bmatrix} A_{D1} \\ A_{D2} \end{bmatrix} \qquad \mathbf{A}_{DL}$$

In general, these matric
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Subtracting $\mathbf{A}_{DL}$ fron
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This equation represent
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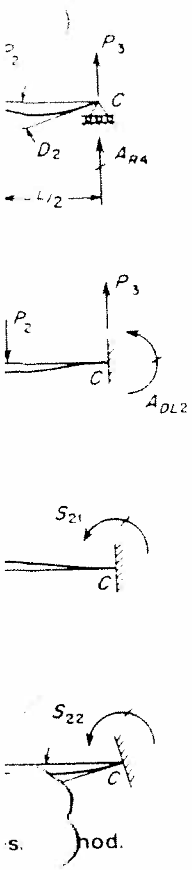

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The condition that D_2 is equal
in Fig. 3-2d. In the figure the
D_1 while the stiffness S_{22} is the
ch case the stiffness coefficient
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now be written. Let the actions in the actual structure corresponding to D_1 and D_2 be denoted A_{D1} and A_{D2}, respectively. These actions will be zero in all cases except when a concentrated external action is applied at a joint corresponding to a degree of freedom. In the example of Fig. 3-2, the action A_{D1} is equal to the couple M while the action A_{D2} is zero. The superposition equations express the fact that the actions in the original structure (Fig. 3-2a) are equal to the corresponding actions in the restrained structure due to the loads (Fig. 3-2b) plus the corresponding actions in the restrained structure due to the unit displacements (Figs. 3-2c and 3-2d) multiplied by the displacements themselves. Therefore, the superposition equations are

$$\begin{aligned} A_{D1} &= A_{DL1} + S_{11}D_1 + S_{12}D_2 \\ A_{D2} &= A_{DL2} + S_{21}D_1 + S_{22}D_2 \end{aligned} \tag{3-4}$$

The sign convention used throughout these equations is that moments are positive when in the same sense (counterclockwise) as the corresponding unknown displacements.

When Eqs. (3-4) are expressed in matrix form, they become

Action (Force, or Moment) applied at the joint, corresponding to unknown displ.

$$\mathbf{A}_D = \mathbf{A}_{DL} + \mathbf{SD} \tag{3-5}$$

In this equation the vector $\mathbf{A}_D$ represents the actions in the original beam corresponding to the unknown joint displacements $\mathbf{D}$; the vector $\mathbf{A}_{DL}$ represents actions in the restrained structure corresponding to the unknown joint displacements and caused by the loads (that is, all loads except those corresponding to the unknown displacements); and $\mathbf{S}$ is the stiffness matrix corresponding to the unknown displacements. The stiffness matrix $\mathbf{S}$ could also be denoted $\mathbf{A}_{DD}$, since it represents actions corresponding to the unknown joint displacements and caused by unit values of those displacements. For the example of Fig. 3-2 the matrices are as follows:

$$\mathbf{A}_D = \begin{bmatrix} A_{D1} \\ A_{D2} \end{bmatrix} \qquad \mathbf{A}_{DL} = \begin{bmatrix} A_{DL1} \\ A_{DL2} \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

In general, these matrices will have as many rows as there are unknown joint displacements. Thus, if d represents the number of unknown displacements, the order of the stiffness matrix $\mathbf{S}$ is $d \times d$, while $\mathbf{A}_D$, $\mathbf{A}_{DL}$, and $\mathbf{D}$ are vectors of order $d \times 1$.

Subtracting $\mathbf{A}_{DL}$ from both sides of Eq. (3-5) and then premultiplying by $\mathbf{S}^{-1}$ gives the following equation for the unknown displacements:

applied joint load — Equivalent Jt. Load

$$\mathbf{D} = \mathbf{S}^{-1}(\mathbf{A}_D - \mathbf{A}_{DL}) \tag{3-6}$$

This equation represents the solution for the displacements in matrix terms because the elements of $\mathbf{A}_D$, $\mathbf{A}_{DL}$, and $\mathbf{S}$ are either known or may be obtained from the restrained structure. Moreover, the member end-actions and reactions for the structure may be found after the joint displacements are known. The procedure for performing such calculations will be illustrated later.

At this point it can be observed that the term $-\mathbf{A}_{DL}$ in Eq. (3-6) represents a vector of *equivalent joint loads*, as described previously in Art. 1.12. Such loads are defined as the negatives of restraint actions corresponding to the unknown joint displacements, and they are caused by loads applied to members of the restrained structure. When these equivalent joint loads are added to the actual joint loads $\mathbf{A}_D$, the results are called *combined joint loads* (see Art. 1.12). Thus, the parentheses in Eq. (3-6) contain combined joint loads for the stiffness method of analysis.

In order to demonstrate the use of Eq. (3-6), the beam in Fig. 3-2a will be analyzed for the values of the loads previously given as

$$P_1 = 2P \qquad M = PL \qquad P_2 = P \qquad P_3 = P$$

When the loads P_1, P_2, and P_3 act upon the restrained structure (Fig. 3-2b), the actions A_{DL1} and A_{DL2}, corresponding to D_1 and D_2, respectively, are developed by the restraints at B and C. Since the couple M corresponds to one of the unknown displacements, it is taken into account later by means of the matrix $\mathbf{A}_D$. The actions A_{DL1} and A_{DL2} are found from the formulas for fixed-end moments (see Case 1, Table B-1):

$$A_{DL1} = -\frac{P_1 L}{8} + \frac{P_2 L}{8} = -\frac{PL}{8}$$

$$A_{DL2} = -\frac{P_2 L}{8} = -\frac{PL}{8}$$

Therefore, the matrix $\mathbf{A}_{DL}$ is

$$\mathbf{A}_{DL} = \frac{PL}{8}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

not enter into matrix A_{DL}

It may be observed from these calculations that the load P_3 does not enter into the matrix $\mathbf{A}_{DL}$, hence, it does not affect the calculations for the joint displacements. However, this load does affect the calculations for the reactions of the actual beam, which are given later.

The stiffness matrix $\mathbf{S}$ consists of the stiffness coefficients shown in Figs. 3-2c and 3-2d. Each coefficient is a couple corresponding to one of the unknown displacements and due to a unit value of one of the displacements. Elements of the first column of the stiffness matrix are shown in Fig. 3-2c, and elements of the second column in Fig. 3-2d. In order to find these coefficients, consider first the fixed-end beam shown in Fig. 3-3. This

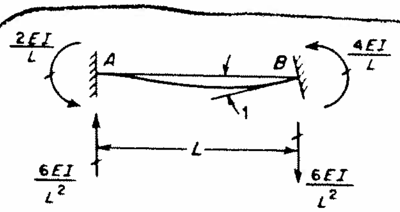

Fig. 3-3. Member stiffnesses for a beam member.

beam is subjected to a unit
developed at end B is $4EI/$
(see Case 3, Table B-4). T
each equal to $6EI/L^2$, and
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The task of calculating
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The stiffness S_{21} is the mo
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Both S_{11} and S_{21} are p
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beam member.

beam is subjected to a unit rotation at end B, and, as a result, the moment developed at end B is $4EI/L$ while the moment at the opposite end is $2EI/L$ (see Case 3, Table B-4). The reactive forces at the ends of the beam are each equal to $6EI/L^2$, and are also indicated in the figure. All of the actions shown in Fig. 3-3 are called *member stiffnesses*, because they are actions at the ends of the member due to a unit displacement of one end. The end of the beam which undergoes the unit displacement is sometimes called the *near end* of the beam, and the opposite end is called the *far end*. Thus, the member stiffnesses at ends A and B are sometimes referred to as the stiffnesses at the far and near ends of the beam. The subject of member stiffnesses is dealt with more extensively in Art. 3.5 and in Chapter 4.

The task of calculating the joint stiffnesses S_{11} and S_{21} in Fig. 3-2c may now be performed through the use of member stiffnesses. When the beam is rotated through a unit angle at joint B, a moment equal to $4EI/L$ is developed at B because of the rotation of the end of member AB. Also, a moment equal to $4EI/L$ is developed at B because of the rotation of the end of member BC. Thus, the total moment at B, equal to S_{11}, is

$$S_{11} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{8EI}{L} \tag{a}$$

The stiffness S_{21} is the moment developed at joint C when joint B is rotated through a unit angle. Since joint C is at the far end of the member, the stiffness coefficient is

$$S_{21} = \frac{2EI}{L}$$

Both S_{11} and S_{21} are positive because they act in the counterclockwise sense. The stiffness coefficients S_{12} and S_{22} are shown in Fig. 3-2d. The first of these is equal to $2EI/L$, since it is an action at the far end of the member BC, while the latter is equal to $4EI/L$ since it is at the near end of the member.

The stiffness matrix $\mathbf{S}$ can be formed using the stiffness coefficients described above:

$$\mathbf{S} = \frac{EI}{L}\begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

Each of the elements in $\mathbf{S}$ is a *joint stiffness*, inasmuch as it represents the action at one of the joints of the structure due to a unit value of a displacement at one of the joints. In this example, the joint stiffness S_{11} (see Fig. 3-2c) is the sum of the near-end member stiffnesses (see Eq. a) for the two members meeting at the joint. Similarly, the stiffness S_{22} is a near-end member stiffness. On the other hand, the stiffnesses S_{12} and S_{21} consist of far-end member stiffnesses for members which connect to a joint that is rotated. In a more general example, it will be found that stiffness elements on the principal diagonal are always composed of near-end stiffnesses while those off the diagonal may consist of either far-end or near-end stiffnesses, as will be seen in later examples. After the stiffness matrix $\mathbf{S}$ is determined, its inverse can be found:

$$\mathbf{S}^{-1} = \frac{L}{14EI}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

The next matrix to be determined is the matrix $\mathbf{A}_D$, representing the actions in the actual structure corresponding to the unknown displacements. In this example the external load which corresponds to the rotation D_1 is the couple M (equal to PL) at joint B. There is no moment at joint C corresponding to D_2, and therefore the matrix $\mathbf{A}_D$ is

$$\mathbf{A}_D = \begin{bmatrix} PL \\ 0 \end{bmatrix}$$

Now that the matrices $\mathbf{A}_D$, $\mathbf{S}^{-1}$, and $\mathbf{A}_{DL}$ have been obtained, the matrix of displacements $\mathbf{D}$ in the actual structure can be found by substituting them into Eq. (3-6), as follows:

$$\mathbf{D} = \frac{L}{14EI}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\left\{\begin{bmatrix} PL \\ 0 \end{bmatrix} - \frac{PL}{8}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\} = \frac{PL^2}{112EI}\begin{bmatrix} 17 \\ -5 \end{bmatrix}$$

Thus, the rotations D_1 and D_2 at joints B and C are

$$D_1 = \frac{17PL^2}{112EI} \qquad D_2 = -\frac{5PL^2}{112EI} \qquad \text{(b)}$$

These results agree with the joint displacements found by the flexibility method in the example of Art. 2.5.

The next step after finding the joint displacements is to determine the member end-actions and the reactions for the structure. As in the flexibility method, there are two approaches that can be followed when performing the calculations by hand. One approach is to obtain the end-actions and reactions by making separate calculations after the joint displacements have been found. The other approach is to set up the calculations in a systematic manner in parallel with the calculations for finding the displacements. In this chapter only the second approach will be utilized.

In order to show how the calculations are performed, consider again the two-span beam shown in Fig. 3-2. As in the flexibility method, the matrices of member end-actions and reactions in the actual structure (Fig. 3-2a) will be denoted $\mathbf{A}_M$ and $\mathbf{A}_R$, respectively. In the restrained structure subjected to the loads (Fig. 3-2b), the matrices of end-actions and reactions corresponding to $\mathbf{A}_M$ and $\mathbf{A}_R$ will be denoted $\mathbf{A}_{ML}$ and $\mathbf{A}_{RL}$, respectively. It should be noted again that when any reference is made to the loads on the restrained structure, it is assumed that all of the actual loads are taken into account except those that correspond to an unknown displacement. Thus,

the joint load M shown in Fig.
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restrained beam in Fig. 3-2b.
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The superposition equatio
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The above two equations and
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When this is done, all joint di
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Consider now the use of E
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matrices $\mathbf{A}_{ML}$, $\mathbf{A}_{RL}$, $\mathbf{A}_{MD}$, and
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$$\left.\begin{matrix}1\\1\end{matrix}\right]\Bigg\} = \frac{PL^2}{112EI}\begin{bmatrix}17\\-5\end{bmatrix}$$

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$$\frac{5}{112EI} \qquad \text{(b)}$$

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the joint load M shown in Fig. 3-2a does not appear on the restrained structure in Fig. 3-2b. However, all other loads are considered to act on the restrained beam in Fig. 3-2b, including the load P_3. This load does not affect the end-actions A_{ML} in the restrained structure, but the reactions A_{RL} are affected. Each of the matrices $\mathbf{A_M}$ and $\mathbf{A_{ML}}$ is of order $m \times 1$, assuming that m represents the number of member end-actions to be found. Similarly, the matrices $\mathbf{A_R}$ and $\mathbf{A_{RL}}$ are of order $r \times 1$, in which r denotes the number of reactions to be found.

In the restrained structure subjected to unit displacements (Figs. 3-2c and 3-2d), the matrices of end-actions and reactions will be denoted $\mathbf{A_{MD}}$ and $\mathbf{A_{RD}}$, respectively. The first column of each of the matrices will contain the actions obtained from the restrained beam in Fig. 3-2c while the second column is made up of actions obtained from the beam in Fig. 3-2d. In the general case the matrices $\mathbf{A_{MD}}$ and $\mathbf{A_{RD}}$ are of order $m \times d$ and $r \times d$, respectively, in which d represents the number of unknown displacements.

The superposition equations for the end-actions and reactions in the actual structure may now be expressed in matrix form:

$$\mathbf{A_M} = \mathbf{A_{ML}} + \mathbf{A_{MD}D} \tag{3-7}$$

$$\mathbf{A_R} = \mathbf{A_{RL}} + \mathbf{A_{RD}D} \tag{3-8}$$

The above two equations and Eq. (3-5) together constitute the three action superposition equations of the stiffness method. The complete solution of a structure consists of solving for the matrix $\mathbf{D}$ of displacements from Eq. (3-6) and then substituting into Eqs. (3-7) and (3-8) to determine $\mathbf{A_M}$ and $\mathbf{A_R}$. When this is done, all joint displacements, member end-actions, and reactions for the structure will be known.

Consider now the use of Eqs. (3-7) and (3-8) in the solution of the two-span beam shown in Fig. 3-2. The unknown displacements $\mathbf{D}$ have already been found (see Eqs. b), and all that remains is the determination of the matrices $\mathbf{A_{ML}}$, $\mathbf{A_{RL}}$, $\mathbf{A_{MD}}$, and $\mathbf{A_{RD}}$. Assume that the member end-actions to be calculated are the shearing force A_{M1} and moment A_{M2} at end B of member AB, and the shearing force A_{M3} and moment A_{M4} at end B of member BC. These are the same end-actions considered previously in the solution by the flexibility method (see Fig. 2-11b), and are selected solely for illustrative purposes. Also, assume that the reactions to be calculated are the force A_{R1} and couple A_{R2} at support A, and the forces A_{R3} and A_{R4} at supports B and C (see Fig. 3-2a). The first two of these reactions are the same as in the earlier solution, and the last two are the redundants from the earlier solution. All of these actions are assumed positive when either upward or counterclockwise.

In the restrained structure subjected to the loads (Fig. 3-2b), the end-actions and reactions in terms of the loads P_1, P_2, and P_3 are seen to be as follows:

$$A_{ML1} = \frac{P_1}{2} \qquad A_{ML2} = -\frac{P_1 L}{8} \qquad A_{ML3} = \frac{P_2}{2} \qquad A_{ML4} = \frac{P_2 L}{8}$$

$$A_{RL1} = \frac{P_1}{2} \qquad A_{RL2} = \frac{P_1 L}{8} \qquad A_{RL3} = \frac{P_1}{2} + \frac{P_2}{2} \qquad A_{RL4} = \frac{P_2}{2} - P_3$$

The values of the loads ($P_1 = 2P, P_2 = P, P_3 = P$) can now be substituted into these expressions, after which the matrices $\mathbf{A}_{ML}$ and $\mathbf{A}_{RL}$ can be formed:

$$\mathbf{A}_{ML} = \frac{P}{8}\begin{bmatrix} 8 \\ -2L \\ 4 \\ L \end{bmatrix} \qquad \mathbf{A}_{RL} = \frac{P}{4}\begin{bmatrix} 4 \\ L \\ 6 \\ -2 \end{bmatrix}$$

The matrices $\mathbf{A}_{MD}$ and $\mathbf{A}_{RD}$ are obtained from an analysis of the beams shown in Figs. 3-2c and 3-2d. For example, the member end-action A_{MD11} is the shearing force at end B of member AB due to a unit displacement corresponding to D_1 (Fig. 3-2c). Thus, this end-action is

$$A_{MD11} = -\frac{6EI}{L^2}$$

as can be seen from Fig. 3-3. The reaction A_{RD11} is the vertical force at support A in the beam of Fig. 3-2c, and is

$$A_{RD11} = \frac{6EI}{L^2}$$

In a similar manner, the other member end-actions and reactions can be found for the beam shown in Fig. 3-2c. These quantities constitute the first columns of the matrices $\mathbf{A}_{MD}$ and $\mathbf{A}_{RD}$. The terms in the second columns are found by similar analyses that are made for the beam shown in Fig. 3-2d. The results are as follows:

$$\mathbf{A}_{MD} = \frac{EI}{L^2}\begin{bmatrix} -6 & 0 \\ 4L & 0 \\ 6 & 6 \\ 4L & 2L \end{bmatrix} \qquad \mathbf{A}_{RD} = \frac{EI}{L^2}\begin{bmatrix} 6 & 0 \\ 2L & 0 \\ 0 & 6 \\ -6 & -6 \end{bmatrix}$$

Substituting the matrices $\mathbf{A}_{ML}$ and $\mathbf{A}_{MD}$ given above, as well as the matrix $\mathbf{D}$ obtained earlier, into Eq. (3-7) gives the following:

$$\mathbf{A}_M = \frac{P}{8}\begin{bmatrix} 8 \\ -2L \\ 4 \\ L \end{bmatrix} + \frac{EI}{L^2}\begin{bmatrix} -6 & 0 \\ 4L & 0 \\ 6 & 6 \\ 4L & 2L \end{bmatrix}\frac{PL^2}{112EI}\begin{bmatrix} 17 \\ -5 \end{bmatrix} = \frac{P}{56}\begin{bmatrix} 5 \\ 20L \\ 64 \\ 36L \end{bmatrix}$$

These results agree with those found previously by the flexibility method

(see Art. 2.5). By substitu
the reactions are found to l

These results also agree
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The method of solutio
3-2 is quite general in its l
(3-8) may be used in the s
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3.3 Examples. Th
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$$A_{ML4} = \frac{P_2 L}{8}$$

$$A_{RL4} = \frac{P_2}{2} - P_3$$

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$$\frac{P}{4}\begin{bmatrix} 4 \\ L \\ 6 \\ -2 \end{bmatrix}$$

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beam shown in Fig. 3-2d.

$$\frac{EI}{L^2}\begin{bmatrix} 6 & 0 \\ 2L & 0 \\ 0 & 6 \\ -6 & -6 \end{bmatrix}$$

iven above, as well as the
he following:

$$\frac{}{EI}\begin{bmatrix} 17 \\ -5 \end{bmatrix} = \frac{P}{56}\begin{bmatrix} 5 \\ 20L \\ 64 \\ 36L \end{bmatrix}$$

sly by the flexibility method

(see Art. 2.5). By substituting the matrices $\mathbf{A}_{RL}$, $\mathbf{A}_{RD}$, and $\mathbf{D}$ into Eq. (3-8) the reactions are found to be:

$$\mathbf{A}_R = \frac{P}{56}\begin{bmatrix} 107 \\ 31L \\ 69 \\ -64 \end{bmatrix}$$

These results also agree with those obtained previously by the flexibility method.

The method of solution described above for the two-span beam in Fig. 3-2 is quite general in its basic concepts, and the matrix equations (3-5) to (3-8) may be used in the solution of any type of framed structure. Also, the equations apply to structures having any number of degrees of kinematic indeterminacy n, in which case the order of the stiffness matrix $\mathbf{S}$ will be $n \times n$. Several examples illustrating the stiffness method are given in the following article.

3.3 Examples. The examples presented in this article illustrate the application of the stiffness method to several types of structures. In each example the object of the calculations is to determine the unknown joint displacements and certain selected member end-actions and reactions. Since the number of degrees of kinematic indeterminacy is small, the problems are suitable for solution by hand. All of the examples are solved in literal form in order to show clearly how the various terms in the matrices are obtained.

Example 1. The three-span continuous beam shown in Fig. 3-4a has fixed supports at A and D and roller supports at B and C; the length of the middle span is equal to 1.5 times the length of each end span. The loads on the beam are assumed to be two concentrated forces acting downward at the positions shown, a uniform load of intensity w acting on spans BC and CD, and a clockwise couple M applied at joint C. All members of the beam are assumed to have the same flexural rigidity EI.

The unknown joint displacements for the beam are the rotations at supports B and C, denoted D_1 and D_2, respectively, as shown in Fig. 3-4b. For illustrative purposes in this example, it will be assumed that the only member end-actions to be determined are the shearing force A_{M1} and the moment A_{M2} at the left-hand end of member AB, and the shearing force A_{M3} and moment A_{M4} at the left-hand end of member BC, as shown in Fig. 3-4b. The reactions to be found in this example are the vertical forces A_{R1} and A_{R2} at supports B and C, respectively. Other reactions could also be obtained if desired. All of the end-actions, reactions, and joint displacements are assumed to be positive when upward or counterclockwise.

The only load on the structure that corresponds to one of the unknown joint displacements is the couple M, which corresponds to the rotation D_2 (except that it is in the opposite sense). Therefore, the vector $\mathbf{A}_D$ of actions corresponding to the unknown displacements is

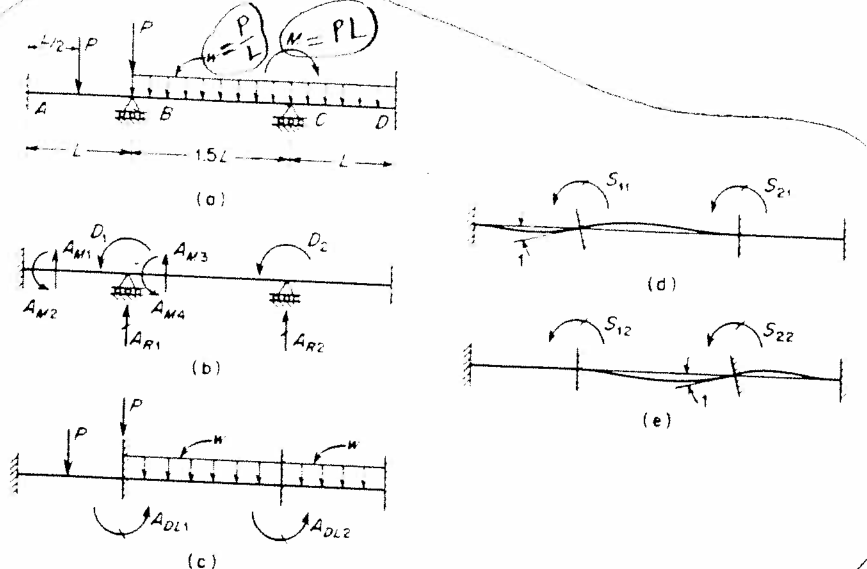

Fig. 3-4. Example 1: Continuous beam.

$$\mathbf{A}_D = \begin{bmatrix} 0 \\ -M \end{bmatrix}$$

The remaining loads on the beam are taken into account by considering them to act on the restrained structure shown in Fig. 3-4c. This structure consists of fixed-end beams and is obtained by preventing joints B and C from rotating. The actions A_{DL1} and A_{DL2} exerted on the beam by the artificial restraints are the couples at supports B and C. Each of the couples is an action corresponding to a displacement D and caused by loads on the beam. The couples can be evaluated without difficulty by referring to the formulas for fixed-end moments given in Table B-1 of Appendix B (see Cases 1 and 6):

$$A_{DL1} = -\frac{PL}{8} + \frac{w(1.5L)^2}{12} = -\frac{PL}{8} + \frac{3wL^2}{16}$$

$$A_{DL2} = -\frac{w(1.5L)^2}{12} + \frac{wL^2}{12} = -\frac{5wL^2}{48}$$

Thus, the vector $\mathbf{A}_{DL}$ becomes

$$\mathbf{A}_{DL} = \frac{L}{48}\begin{bmatrix} -6P + 9wL \\ -5wL \end{bmatrix}$$

The end-actions A_{ML} and the reactions A_{RL} for the restrained beam of Fig. 3-4c can be determined also by referring to the table of fixed-end actions. For example, the end-action A_{ML1} is the shearing force at the left-hand end of member AB and is equal to $P/2$; similarly, the reaction A_{RL1} is the vertical reaction at support B, obtained as follows:

$$A_{RL1} = \frac{P}{2} + P + \frac{w(1.5L)}{2} = \frac{3P}{2} + \frac{3wL}{4}$$

By continuing in the same man
ture can be found. The resultin

$\mathbf{A}_{ML}$ =

Note that the load P acting
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In order to simplify the sub
relationships exist between the

Substitution of these relations
yields

$\mathbf{A}_D = PL\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\mathbf{A}_{DL}$ =

The next step in the solutio
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taken into account are unit ro
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elements of the stiffness matri
stiffnesses can be found witho

S_{11} =

S_{12} =

Therefore, the stiffness matrix

and the inverse matrix is

The joint displacements may n
$\mathbf{A}_{DL}$ into Eq. (3-6) and evaluati

$$\mathbf{D} = \frac{L}{32EI}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

By continuing in the same manner, all of the required actions in the restrained structure can be found. The resulting vectors $\mathbf{A}_{ML}$ and $\mathbf{A}_{RL}$ are:

$$\mathbf{A}_{ML} = \begin{bmatrix} \dfrac{P}{2} \\ \dfrac{PL}{8} \\ \dfrac{3wL}{4} \\ \dfrac{3wL^2}{16} \end{bmatrix} \qquad \mathbf{A}_{RL} = \begin{bmatrix} \dfrac{3P}{2} + \dfrac{3wL}{4} \\ \dfrac{5wL}{4} \end{bmatrix}$$

Note that the load P acting downward at joint B affects the reactions in the restrained beam but not the member end-actions.

In order to simplify the subsequent calculations, assume now that the following relationships exist between the various loads on the beam:

$$wL = P \qquad M = PL$$

Substitution of these relations into the matrices given in the preceding paragraphs yields

$$\mathbf{A}_D = PL\begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \mathbf{A}_{DL} = \frac{PL}{48}\begin{bmatrix} 3 \\ -5 \end{bmatrix} \qquad \mathbf{A}_{ML} = \frac{P}{16}\begin{bmatrix} 8 \\ 2L \\ 12 \\ 3L \end{bmatrix} \qquad \mathbf{A}_{RL} = \frac{P}{4}\begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

The next step in the solution is the analysis of the restrained beam for the effects of unit displacements corresponding to the unknowns. The two conditions to be taken into account are unit rotations at joints B and C, as illustrated in Figs. 3-4d and 3-4e. The four couples acting at joints B and C in these figures represent the elements of the stiffness matrix $\mathbf{S}$. With the formulas given in Fig. 3-3, each of these stiffnesses can be found without difficulty, as shown in the following calculations:

$$S_{11} = S_{22} = \frac{4EI}{L} + \frac{4EI}{1.5L} = \frac{20EI}{3L}$$

$$S_{12} = S_{21} = \frac{2EI}{1.5L} = \frac{4EI}{3L}$$

Therefore, the stiffness matrix $\mathbf{S}$ becomes

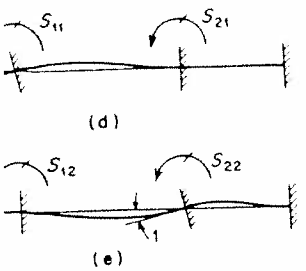
$$\mathbf{S} = \frac{4EI}{3L}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

and the inverse matrix is

$$\mathbf{S}^{-1} = \frac{L}{32EI}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

The joint displacements may now be found by substituting the matrices $\mathbf{S}^{-1}$, $\mathbf{A}_D$, and $\mathbf{A}_{DL}$ into Eq. (3-6) and evaluating $\mathbf{D}$, as follows:

$$\mathbf{D} = \frac{L}{32EI}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}\left\{PL\begin{bmatrix} 0 \\ -1 \end{bmatrix} - \frac{PL}{48}\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\} = \frac{PL^2}{384EI}\begin{bmatrix} 7 \\ -53 \end{bmatrix}$$



...unt by considering them to act ...structure consists of fixed-end ... rotating. The actions A_{DL1} ...nts are the couples at supports ...nding to a displacement D and ... evaluated without difficulty by ...en in Table B-1 of Appendix B

$$\frac{PL}{8} + \frac{3wL^2}{16}$$
$$\frac{5wL^2}{48}$$

L]

...the restrained beam of Fig. 3-4c ...fixed-end actions. For example, ...-hand end of member AB and is ...vertical reaction at support B.

$$\frac{3P}{2} + \frac{3wL}{4}$$

This matrix gives the rotations at joints B and C of the continuous beam shown in Fig. 3-4a.

For the determination of the member end-actions and the reactions, it is necessary to consider again the restrained beams shown in Figs. 3-4d and 3-4e. In each of these beams there are end-actions and reactions that correspond to the end-actions and reactions selected previously and shown in Fig. 3-4b. These quantities for the beams with unit displacements are denoted A_{MD} and A_{RD}, respectively. For example, the end-action A_{MD11} is the shearing force at the left-hand end of member AB due to a unit value of D_1 (see Fig. 3-4d). The end-action A_{MD21} is the moment at the same location. In all cases the first subscript identifies the end-action itself and the second signifies the unit displacement that produces the action. The reactions in the beams of Figs. 3-4d and 3-4e follow a similar pattern, with A_{RD11} and A_{RD21} being the reactions at supports B and C, respectively, due to a unit value of the displacement D_1 (Fig. 3-4d). With this identification scheme in mind, and also using the formulas given in Fig. 3-3, it is not difficult to calculate the various end-actions and reactions. For example, the end-actions in the beam of Fig. 3-4d are the following:

OK

$$A_{MD11} = \frac{6EI}{L^2} \qquad A_{MD21} = \frac{2EI}{L} \qquad A_{MD31} = \frac{6EI}{(1.5L)^2} = \frac{8EI}{3L^2} \qquad A_{MD41} = \frac{4EI}{1.5L} = \frac{8EI}{3L}$$

Also, the reactions in the same beam are

$$A_{RD11} = -\frac{6EI}{L^2} + \frac{6EI}{(1.5L)^2} = -\frac{10EI}{3L^2} \qquad A_{RD21} = -\frac{6EI}{(1.5L)^2} = -\frac{8EI}{3L^2}$$

Similarly, the end-actions and reactions for the beam in Fig. 3-4e can be found, after which the matrices $\mathbf{A_{MD}}$ and $\mathbf{A_{RD}}$ are constructed. These matrices are

$$\mathbf{A_{MD}} = \frac{2EI}{3L^2}\begin{bmatrix} 9 & 0 \\ 3L & 0 \\ 4 & 4 \\ 4L & 2L \end{bmatrix} \qquad \mathbf{A_{RD}} = \frac{2EI}{3L^2}\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$$

The final steps in the solution consist of calculating the matrices $\mathbf{A_M}$ and $\mathbf{A_R}$ for the end-actions and reactions in the original beam of Fig. 3-4a. These matrices are obtained by substituting into Eqs. (3-7) and (3-8) the matrices $\mathbf{D}$, $\mathbf{A_{ML}}$, $\mathbf{A_{MD}}$, $\mathbf{A_{RL}}$, and $\mathbf{A_{RD}}$, all of which have been determined above. The results are

$$\mathbf{A_M} = \frac{P}{576}\begin{bmatrix} 351 \\ 93L \\ 248 \\ 30L \end{bmatrix} \qquad \mathbf{A_R} = \frac{P}{576}\begin{bmatrix} 1049 \\ 427 \end{bmatrix}$$

Thus, all of the selected end-actions and reactions for the beam have been calculated.

Example 2. The continuous beam ABC shown in Fig. 3-5a has a fixed support at A, a roller support at B, and a guided support at C. Therefore, the only unknown joint displacements are the rotation at support B and the vertical translation at support C. These displacements are denoted D_1 and D_2, respectively, as identified in Fig. 3-5b. The beam has constant flexural rigidity EI and is subjected to the loads P_1 and P_2, acting at the positions shown in the figure. It will be assumed that the loads are given as follows:

Fig. 3-5

For illustrative purposes in this
reactions for the beam are to be
ing force A_{M1} and the moment
3-5b), and the reactions are the
vertical force A_{R3} at support B,
actions and displacements are s

The restrained structure is
lation at joint C, thereby giving
the loads on this restrained str
displacements D are

$$A_{DL1} = -\frac{P_1 L}{8}$$

Also, the member end-actions i

$$A_{ML1} = \frac{P_1}{2}$$

continuous beam shown in

...d the reactions, it is neces-
...s. 3-4d and 3-4e. In each of
...rrespond to the end-actions
...4b. These quantities for the
...$_{RD}$, respectively. For exam-
...ft-hand end of member AB
...n A_{MD21} is the moment at the
the end-action itself and the
... action. The reactions in the
..., with A_{RD11} and A_{RD21} being
... a unit value of the displace-
in mind, and also using the
... the various end-actions and
...f Fig. 3-4d are the following:

$$\frac{EI}{L^2} \qquad A_{MD41} = \frac{4EI}{1.5L} = \frac{8EI}{3L}$$

$$\ldots = -\frac{6EI}{(1.5L)^2} = -\frac{8EI}{3L^2}$$

in Fig. 3-4e can be found, after
... trices are

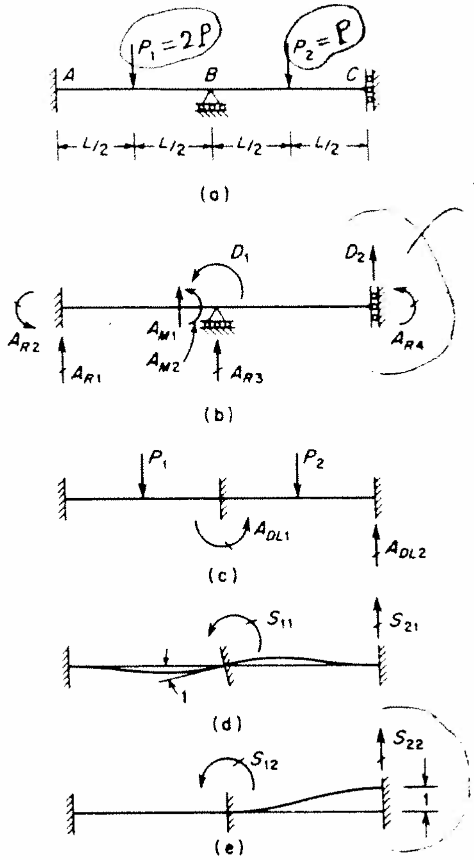
$$\frac{EI}{L^2}\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$$

ing the matrices $\mathbf{A_M}$ and $\mathbf{A_R}$ for
f Fig. 3-4a. These matrices are
matrices $\mathbf{D}$, $\mathbf{A_{ML}}$, $\mathbf{A_{MD}}$, $\mathbf{A_{RL}}$, and
results are

$$\ldots 6 \begin{bmatrix} 1049 \\ 427 \end{bmatrix}$$

ions for the beam have been

...wn in Fig. 3-5a has a fixed sup-
...pport at C. Therefore, the only
...pport B and the vertical transla-
...ted D_1 and D_2, respectively, as
...ural rigidity EI and is subjected
...n in the figure. It will be assumed



This support is NOT a roller, nor fixed. It is a fixed support, with free vertical displ.

Remarks
(a) In the restrained structure, all displ are assumed to be fixed
(b) Actually, there was zero displ D_2. Hence, we apply unit vert. displ. @ jt. C

Fig. 3-5. Example 2: Continuous beam.

$$P_1 = 2P \qquad P_2 = P$$

For illustrative purposes in this example, certain member end-actions and all of the reactions for the beam are to be calculated. The selected end-actions are the shearing force A_{M1} and the moment A_{M2} at the right-hand end of member AB (see Fig. 3-5b), and the reactions are the vertical force A_{R1} and couple A_{R2} at support A, the vertical force A_{R3} at support B, and the couple A_{R4} at support C (see Fig. 3-5b). All actions and displacements are shown in their positive directions in the figure.

The restrained structure is obtained by preventing rotation at joint B and translation at joint C, thereby giving the two fixed-end beams shown in Fig. 3-5c. Due to the loads on this restrained structure, the actions corresponding to the unknown displacements D are

$$A_{DL1} = -\frac{P_1L}{8} + \frac{P_2L}{8} = -\frac{PL}{8} \qquad A_{DL2} = \frac{P_2}{2} = \frac{P}{2}$$

Also, the member end-actions in the same beam are

$$A_{ML1} = \frac{P_1}{2} = P \qquad A_{ML2} = -\frac{P_1L}{8} = -\frac{PL}{4}$$

and the reactions are

$$A_{RL1} = \frac{P_1}{2} = P \qquad A_{RL2} = \frac{P_1 L}{8} = \frac{PL}{4}$$

$$A_{RL3} = \frac{P_1}{2} + \frac{P_2}{2} = \frac{3P}{2} \qquad A_{RL4} = -\frac{P_2 L}{8} = -\frac{PL}{8}$$

From the values given above, the following matrices that are required in the solution can be formed:

$$\mathbf{A}_{DL} = \frac{P}{8}\begin{bmatrix} -L \\ 4 \end{bmatrix} \qquad \mathbf{A}_{ML} = \frac{P}{4}\begin{bmatrix} 4 \\ -L \end{bmatrix} \qquad \mathbf{A}_{RL} = \frac{P}{8}\begin{bmatrix} 8 \\ 2L \\ 12 \\ -L \end{bmatrix}$$

After obtaining the matrices of actions due to loads, the next step is to analyze the restrained beam for unit values of the unknown displacements, as shown in Figs. 3-5d and 3-5e. The stiffnesses S_{11} and S_{21} caused by a unit rotation at joint B are readily obtained from the formulas given in Fig. 3-3, as follows:

$$S_{11} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{8EI}{L} \qquad S_{21} = -\frac{6EI}{L^2}$$

In the case of a unit displacement corresponding to D_2 (Fig. 3-5e), it is necessary to have formulas for the forces and couples at the ends of a fixed-end beam subjected to a translation of one end relative to the other. The required formulas can be obtained from Table B-4 of Appendix B (see Case 2). When the translation is equal to unity, the couples at the ends are equal to $6EI/L^2$, and the forces are equal to $12EI/L^3$, as shown in Fig. 3-6. From these values, the stiffnesses S_{12} and S_{22} for the beam in Fig. 3-5e are seen to be as follows:

$$S_{12} = -\frac{6EI}{L^2} \qquad S_{22} = \frac{12EI}{L^3}$$

Therefore, the stiffness matrix can be constructed, and its inverse obtained:

$$\mathbf{S} = \frac{2EI}{L^3}\begin{bmatrix} 4L^2 & -3L \\ -3L & 6 \end{bmatrix} \qquad \mathbf{S}^{-1} = \frac{L}{30EI}\begin{bmatrix} 6 & 3L \\ 3L & 4L^2 \end{bmatrix}$$

The inverse matrix, as well as the matrix $\mathbf{A}_{DL}$ determined previously, can now be substituted into Eq. (3-6) in order to obtain the matrix $\mathbf{D}$ of unknown displacements. The matrix $\mathbf{A}_D$ appearing in the equation is a null matrix since there are no loads on the original beam corresponding to either D_1 or D_2. The solution for $\mathbf{D}$ is found to be

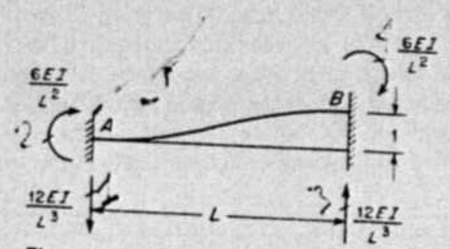


Fig. 3-6. Member stiffnesses for a beam member.

The matrices $\mathbf{A}_{MD}$ and $\mathbf{A}_{RD}$ end-actions and reactions, resp 3-5e. The first column of each m ment D_1 (Fig. 3-5d), and the sec of the elements in these matrice in Figs. 3-3 and 3-6, and the rest

$$\mathbf{A}_{MD} = \frac{2EI}{L^2}\Big[$$

Then the matrices $\mathbf{A}_M$ and $\mathbf{A}_R$ c $\mathbf{A}_{RL}$, $\mathbf{A}_{RD}$, and $\mathbf{D}$ into Eqs. (3-7)

$$\mathbf{A}_M = \frac{P}{2(}$$

Thus, all of the desired membe joint displacements, have been

Example 3. The purpos plane truss by the stiffness meth consists of four members mee selected because it has only twc the horizontal and vertical tran discussion pertaining to the solı cated trusses.

It is a convenience in the ar Therefore, the members are nu circles in Fig. 3-7a. Also, for th that the four members have le EA_2, EA_3, and EA_4, respectivel values in order that the solution

The loads on the truss cons at joint E, as well as the weig distributed loads along the mem and w_4, respectively, for each the weight of the member per u For example, the total weight o

The unknown displacemen taken as the horizontal and ver as well as the applied loads a toward the right or upward. Th as the axial forces in the four r These actions are shown in Fi Because of the weights of the m

$$\frac{{}_1L}{3} = \frac{PL}{4}$$

$$-\frac{P_2L}{8} = -\frac{PL}{8}$$

that are required in the solution

$$\mathbf{A}_{RL} = \frac{P}{8}\begin{bmatrix} 8 \\ 2L \\ 12 \\ -L \end{bmatrix}$$

...ads, the next step is to analyze ...wn displacements, as shown in ...sed by a unit rotation at joint B 3-3, as follows:

$$_{21} = -\frac{6EI}{L^2}$$

D_2 (Fig. 3-5e), it is necessary to ...ls of a fixed-end beam subjected The required formulas can be 2). When the translation is equal ...$/L^2$, and the forces are equal to ...h... stiffnesses S_{12} and S_{22} for the

$$\frac{12EI}{L^3}$$

, and its inverse obtained:

$$\frac{L}{30EI}\begin{bmatrix} 6 & 3L \\ 3L & 4L^2 \end{bmatrix}$$

determined previously, can now ...e matrix $\mathbf{D}$ of unknown displace-...s a null matrix since there are no ...r D_1 or D_2. The solution for $\mathbf{D}$ is

$\frac{6EI}{L^2}$

1

$\frac{2EI}{L^3}$

for a beam member.

$$\mathbf{D} = \frac{PL^2}{240EI}\begin{bmatrix} -6 \\ -13L \end{bmatrix}$$

The matrices $\mathbf{A}_{MD}$ and $\mathbf{A}_{RD}$ which appear in Eqs. (3-7) and (3-8) represent the end-actions and reactions, respectively, in the restrained beams of Figs. 3-5d and 3-5e. The first column of each matrix is associated with a unit value of the displacement D_1 (Fig. 3-5d), and the second column with a unit value of D_2 (Fig. 3-5e). All of the elements in these matrices can be obtained with the aid of the formulas given in Figs. 3-3 and 3-6, and the results are as follows:

$$\mathbf{A}_{MD} = \frac{2EI}{L^2}\begin{bmatrix} -3 & 0 \\ 2L & 0 \end{bmatrix} \qquad \mathbf{A}_{RD} = \frac{2EI}{L^3}\begin{bmatrix} 3L & 0 \\ L^2 & 0 \\ 0 & -6 \\ L^2 & -3L \end{bmatrix}$$

Then the matrices $\mathbf{A}_M$ and $\mathbf{A}_R$ can be found by substituting the matrices $\mathbf{A}_{ML}$, $\mathbf{A}_{MD}$, $\mathbf{A}_{RL}$, $\mathbf{A}_{RD}$, and $\mathbf{D}$ into Eqs. (3-7) and (3-8), producing

$$\mathbf{A}_M = \frac{P}{20}\begin{bmatrix} 23 \\ -7L \end{bmatrix} \qquad \mathbf{A}_R = \frac{P}{20}\begin{bmatrix} 17 \\ 4L \\ 43 \\ 3L \end{bmatrix}$$

Thus, all of the desired member end-actions and support reactions, as well as the joint displacements, have been calculated.

Example 3. The purpose of this example is to illustrate the analysis of a plane truss by the stiffness method. The truss to be solved is shown in Fig. 3-7a and consists of four members meeting at a common joint E. This particular truss is selected because it has only two degrees of freedom for joint displacement, namely, the horizontal and vertical translations at joint E. However, most of the ensuing discussion pertaining to the solution of this truss is also applicable to more complicated trusses.

It is a convenience in the analysis to identify the bars of the truss numerically. Therefore, the members are numbered from 1 to 4 as shown by the numbers in circles in Fig. 3-7a. Also, for the purposes of general discussion it will be assumed that the four members have lengths L_1, L_2, L_3, and L_4, and axial rigidities EA_1, EA_2, EA_3, and EA_4, respectively. Later, all of these quantities will be given specific values in order that the solution may be carried to completion.

The loads on the truss consist of the two concentrated forces P_1 and P_2 acting at joint E, as well as the weights of the members. The weights act as uniformly distributed loads along the members and are assumed to be of intensity w_1, w_2, w_3, and w_4, respectively, for each of the four members. In all cases the intensity w is the weight of the member per unit distance measured along the axis of the member. For example, the total weight of member 1 is w_1L_1.

The unknown displacements at joint E, denoted D_1 and D_2 in Fig. 3-7b, are taken as the horizontal and vertical translations of the joint. These displacements, as well as the applied loads at joint E, will be assumed positive when directed toward the right or upward. The member end-actions to be calculated are selected as the axial forces in the four members at the ends A, B, C, and D, respectively. These actions are shown in Fig. 3-7b and are denoted A_{M1}, A_{M2}, A_{M3}, and A_{M4}. Because of the weights of the members, the axial forces at the other ends (that is, at