Slide 1

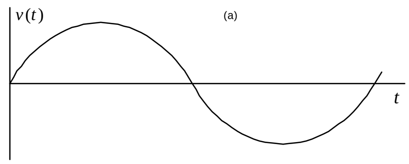
Sampling Theorem

All forms of pulse and digital encoding of analog data involve sampling the analog signal at specific values of time. Thus, one could say that certain portions of the signal are “lost”. The basic question relates to whether or not the signal can be accurately reproduced from these finite samples. The key to the concept is the *sampling theorem*, which will be developed in this module. It serves as the basis for most types of pulse and digital signals, including such common items as music CDs.

Slide 2

Start with a baseband signal.

For convenience, a sinusoidal signal will be shown, but a more complex spectrum will be assumed.



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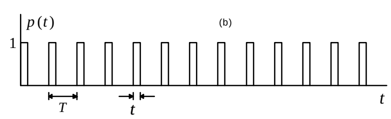
Slide 3

Multiply by a pulse train.



The product of the waveforms is a *sampled-data signal* and is shown on the next slide.

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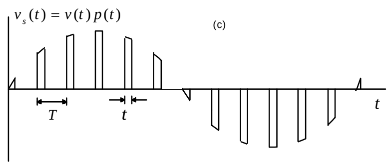
Sampled-Data Signal with Natural Sampling



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Spectrum Development

The sampled signal can be expressed as

$$v_s(t) = v(t)\,p(t)$$

The pulse train can be expressed

in terms of its Fourier series as

$$p(t) = \sum_{-\infty}^{\infty} \overline{P_n} e^{jnw_s t}$$

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Spectrum Development (cont.)

The Fourier coefficients are

$$\bar{P}_n = d\,\frac{\sin np\,d}{np\,d} \qquad d = t\,/T = \text{ duty cycle}$$

$$v_s(t) = v(t)\sum_{-\infty}^{\infty} \bar{P}_n e^{jnw_s t} = \sum_{-\infty}^{\infty} \bar{P}_n\, v(t) e^{jnw_s t}$$

Applying the modulation theorem,

$$\bar{V}_s(f) = \sum_{-\infty}^{\infty} \bar{P}_n \bar{V}(f - nf_s)$$

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Assume the following spectral form for the signal

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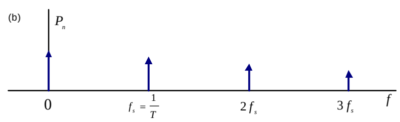
The pulse train has the following spectral form:



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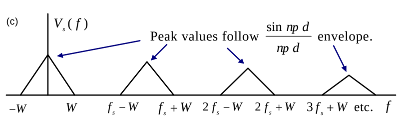
Resulting Spectrum of Sampled-Data Signal



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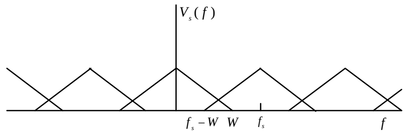
Aliasing results when the sampling rate is too low



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Sampling Theorem

To prevent aliasing, the lowest frequency of the first translated component must be higher than the highest frequency of the baseband signal. If this condition is met, the original signal could theoretically be recovered by low-pass filtering.

$$f_s - W \geq W \quad \text{or}$$

$$f_s \geq 2W$$

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Alternate Form

The time *T* between succesive samples must satisfy

$$T \leq \frac{1}{2W}$$

where

$$T = 1/f_s$$

A convenient definition is the *folding frequency*. It is

$$f_o = f_s/2 = 1/2T$$

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Comments on Sampling Theorem

- While the theoretical development indicates that the minimum sampling rate is $2W$, in practice it needs to be somewhat higher.
- The folding frequency is the highest theoretical frequency that can be reconstructed.
- For a signal of duration t_p, the minimum number of samples N is

$$N = f_s t_p$$

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Example 1. A signal has a spectrum from dc to 5 kHz. Determine minimum sampling rate and maximum time between samples.

$$f_s = 2W = 2\times 5 = 10\ \text{kHz}$$

$$T = \frac{1}{f_s} = \frac{1}{10{,}000} = 1\times 10^{-4}\ \text{s} = 100\ m\text{s}$$

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Example 2. To provide some guard band, assume signal of Example 1 is sampled 25% above theoretical minimum. Repeat the analysis.

$$f_s = 1.25\times 2W = 1.25\times 10\ \text{kHz} = 12.5\ \text{kHz}$$

$$T = \frac{1}{f_s} = \frac{1}{12{,}500} = 80\times 10^{-6}\ \text{s} = 80\ m\text{s}$$

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Example 3. For conditions established in Example 2, assume that the signal lasts for 30 minutes. Determine total number of samples required.

$$N = f_s t_p = 12{,}500 \times 1800$$
$$= 22.5 \times 10^6 \text{ samples}$$

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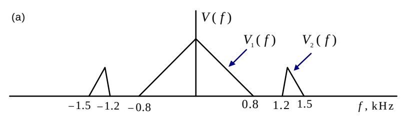
Example 4. Assume that the desired signal below corresponds to V_1 but V_2 is a spurious component. Sketch spectrum for a sampling rate of 2 kHz.



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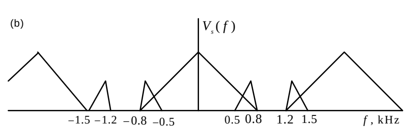
Example 4. The sampled spectrum is shown below. Note how spurious components appear on top of desired signal.



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Example 4. How to resolve the problem.

- Pass the signal through an analog low-pass filter with a cutoff frequency between 800 Hz and 1.2 kHz prior to sampling or
- Sample at a higher rate (greater than 3 kHz).

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Summary

- A baseband signal can theoretically be recovered from samples provided that the sampling rate is equal to or greater than twice the highest frequency.
- The signal may be recovered by passing the sampled signal through an ideal low-pass filter having a cutoff frequency equal to the *folding frequency* (half the sampling frequency.
- In practice, the sampling rate should be greater than the theoretical minimum in order to ease recovery filtering.