Outline

- Taylor series and Maclaurin series.
- Taylor Polynomials
- Review

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^k(a)}{k!}(x-a)^k + \ldots \end{aligned} \tag{1}$$

The Maclaurin series generated by f is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^k(0)}{k!}x^k \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^k(0)}{k!}x^k + \ldots \end{aligned} \tag{2}$$

the Taylor series generated by f at $x = 0$.

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k \tag{3}$$

Ex: Find the first 5 terms of the Taylor series for $f(x) = \cos x$ centered at $a = 2$.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k \tag{4}$$

$$\begin{aligned}
f(x) &= \cos x \quad f(2) = \cos 2 \\
f'(x) &= -\sin x \quad f'(2) = -\sin 2 \\
f''(x) &= -\cos x \quad f''(2) = -\cos 2 \\
f^{(3)}(x) &= \sin x \quad f^{(3)}(2) = \sin 2 \\
f^{(4)}(x) &= \cos x \quad f^{(4)}(2) = \cos 2 \\
f^{(5)}(x) &= -\sin x \quad f^{(5)}(2) = -\sin 2
\end{aligned} \tag{5}$$

$$\begin{aligned}
f(x) = {} & \cos 2 - \sin 2(x-2) - \frac{\cos 2}{2!}(x-2)^2 + \frac{\sin 2}{3!}(x-2)^3 \\
& - \frac{\cos 2}{4!}(x-2)^4 - \frac{\sin 2}{3!}(x-2)^5 + \dots
\end{aligned} \tag{6}$$

Find out Taylor polynomials of order 0,1,2 and 3.

$$\begin{aligned}
P_0(x) &= \cos 2 \\
P_1(x) &= \cos 2 - \sin 2(x-2) \\
P_2(x) &= \cos 2 - \sin 2(x-2) - \frac{\cos 2}{2!}(x-2)^2 \\
P_3(x) &= \cos 2 - \sin 2(x-2) - \frac{\cos 2}{2!}(x-2)^2 + \frac{\sin 2}{3!}(x-2)^3
\end{aligned} \tag{7}$$

Determine whether the sequence converges or diverges. If it converges, find the limit.

- $$a_n = \frac{7-2n^2}{2n+5n^3} \tag{8}$$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{7-2n^2}{2n+5n^3} \\ &= \lim_{n\to\infty} \frac{n^2(\frac{7}{n^2}-2)}{n^3(\frac{2}{n^2}+5)} \\ &= \lim_{n\to\infty} \frac{1}{n}(\frac{\frac{7}{n^2}-2}{\frac{2}{n^2}+5}) = 0 \end{aligned} \tag{9}$$

- $$a_n = \frac{3^n}{4^{n+1}} \tag{10}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{3}{4})^n \frac{1}{4} = 0 \tag{11}$$

Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=0}^{\infty} \frac{3^n + 4^n}{6^n} \tag{12}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{3^n + 4^n}{6^n} &= \sum_{k=0}^{\infty} \frac{3^n}{6^n} + \sum_{k=0}^{\infty} \frac{4^n}{6^n} \\ &= \sum_{k=0}^{\infty} (\frac{1}{2})^n + \sum_{k=0}^{\infty} (\frac{2}{3})^n = \frac{1}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{2}{3}} = 2 + 3 = 5 \end{aligned} \tag{13}$$

Determine whether the series converges or diverges.

- $$\sum_{n=0}^{\infty} \frac{3+\sin n}{n\sqrt{n}} \tag{14}$$

  $$\frac{3+\sin n}{n\sqrt{n}} \leq \frac{4}{n^{\frac{3}{2}}} \tag{15}$$

  Since $\sum_{k=0}^{\infty} \frac{4}{n^{\frac{3}{2}}}$ converges, $\sum_{k=0}^{\infty} \frac{3+\sin n}{n\sqrt{n}}$ converges.

- $$\sum_{n=0}^{\infty} \frac{3^n}{2+4^n} \tag{16}$$

  $$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{2+4^{n+1}} \frac{2+4^n}{3^n} \\ &= \lim_{n\to\infty} 3\frac{2+4^n}{2+4^{n+1}} = \lim_{n\to\infty} 3\frac{4^n(\frac{2}{4^n}+1)}{4^{n+1}(\frac{2}{4^{n+1}}+1)} = \frac{3}{4} < 1 \end{aligned} \tag{17}$$

  Converges

Determine whether the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \tag{18}$$

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tag{19}$$

Diverges for $p-$ series with $p < 1$.
It is an alternating series.

- $u_n > 0$
- $u_n = \frac{1}{\sqrt{n}}$, $u_{n+1} = \frac{1}{\sqrt{n+1}}$, $u_n \geq u_{n+1}$.
- $\lim_{n \to \infty} = 0$

It converges conditionally.

Find out the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \frac{n3^n}{(3x-2)^n}\right| \\ &= |3x-2| \lim_{n\to\infty} \left|\frac{n}{3(n+1)}\right| = \frac{|3x-2|}{3} < 1 \end{aligned} \tag{20}$$

$$|x - \frac{2}{3}| < 1 \quad -\frac{1}{3} < x < \frac{5}{3} \tag{21}$$

$$R = 1 \tag{22}$$

$$x = \frac{5}{3}, \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{diverges} \tag{23}$$

$$x = -\frac{1}{3}, \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{converges} \tag{24}$$

Interval of convergence is $-\frac{1}{3} \leq x < \frac{5}{3}$.