Outline

- Integration of Rational Functions by Partial Fraction

- Rational function is a quotient of polynomials. $\frac{f(x)}{g(x)}$
- Method of partial fractions: Write a rational function as a sum of simpler fractions.

The method of partial fractions:

- The fraction must be proper. The degree of $f(x)$ must be less than the degree of $g(x)$. (the degree of polynomial is the highest order of the polynomial). If not, divide $f(x)$ by $g(x)$. Work with the remainder.
- Find the factors of $g(x)$. Any polynomial with real coefficients can be written as a product of linear factors and real quadratic factors.

The method of partial fractions (cont.):

- Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$. Assign the sume of m partial fractions $\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+...+\frac{A_m}{(x-r)^m}$.
- Let x^2+px+q be a quadratic factor of $g(x)$. Suppose that $(x^2+px+q)^n$ is the highest power of this factor. (x^2+px+q has no real roots). Assign n partial fractions: $\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+...+\frac{B_nx+C_n}{(x^2+px+q)^n}$
- Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
- Equate the coefficients of corresponding powers x and solve for the undetermined coefficients.

Ex1:

$$\int \frac{5x-3}{x^2-2x-3}dx \tag{1}$$

The fraction is proper.
Find the factor of x^2-2x-3. They are $x-3$ and $x+1$.
For $x-3$, it has the partial fraction: $\frac{A}{x-3}$, for $x+1$, it has the partial fraction: $\frac{B}{x+1}$
Set $\frac{5x-3}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$, clear it $\frac{5x-3}{x^2-2x-3} = \frac{(A+B)x+(A-3B}{(x-3)(x+1)}$.
Equate the coefficients of corresponding powers: $A+B=5$ and $A-3B=-3$. Solve it. $A=3, B=2$
So $\frac{5x-3}{x^2-2x-3} = \frac{3}{x-3} + \frac{2}{x+1}$

$$\int \frac{5x-3}{x^2-2x-3}dx = \int \frac{3}{x-3} + \frac{2}{x+1}dx = 3\ln|x-3| + 2\ln|x+1| + C \tag{2}$$

Ex2:

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx \tag{3}$$

The fraction is improper.Long division: $2x + \frac{5x-3}{x^2-2x-3}$, we will work with the remainder $\frac{5x-3}{x^2-2x-3}$

Find the factor of $x^2 - 2x - 3$. They are $x - 3$ and $x + 1$.

For $x - 3$, it has the partial fraction: $\frac{A}{x-3}$, for $x + 1$, it has the partial fraction: $\frac{B}{x+1}$

Set $\frac{5x-3}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$, clear it $\frac{5x-3}{x^2-2x-3} = \frac{(A+B)x+(A-3B}{(x-3)(x+1)}$.

Equate the coefficients of corresponding powers: $A + B = 5$ and $A - 3B = -3$. Solve it. $A = 3, B = 2$

So $\frac{5x-3}{x^2-2x-3} = \frac{3}{x-3} + \frac{2}{x+1}$

$$\int \frac{5x - 3}{x^2 - 2x - 3} dx = \int \frac{3}{x - 3} + \frac{2}{x + 1} dx = 3\ln|x-3| + 2\ln|x+1| + C \tag{4}$$

Ex3:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx \tag{5}$$

The fraction is proper.
The factors are x^2+1 and $(x-1)^2$.
For x^2+1, it has the partial fraction: $\frac{Ax+B}{x^2+1}$, for $(x-1)^2$, it has the partial fraction: $\frac{C}{x-1} + \frac{D}{(x-1)^2}$
Set $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$, clear it
$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{(A+C)x^3+(-2A+B-C+D)x^2+(A-2B+C)x+(B-C+D)}{(x^2+1)(x-1)^2}$.
Equate the coefficients of corresponding powers: $A+C=0$, $-2A+B-C+D=0$, $A-2B+C=-2$ and $B-C+D=4$. Solve it. $A=2, B=1, C=-2, D=1$
So $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}$

$$\begin{aligned}
&\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx = \int \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}dx \\
&= \int \frac{2x}{x^2+1}dx|_{u=x^2+1,du=2x} + \int \frac{dx}{x^2+1} - \\
&2\int \frac{dx}{x-1} + \int \frac{1}{(x-1)^2}dx|_{u=x-1,du=dx} \\
&= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C
\end{aligned} \tag{6}$$

Ex4:

$$\int \frac{dx}{x(x^2+1)^2} \tag{7}$$

The fraction is proper.
The factors are x and $(x^2+1)^2$.
For x, it has the partial fraction: $\frac{A}{x}$, for $(x^2+1)^2$, it has the partial fraction: $\frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$
Set $\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Clear it
$\frac{1}{x(x^2+1)^2} = \frac{(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A}{x(x^2+1)^2}$ Equate the coefficients of corresponding powers: $A+B=0$, $C=0$, $2A+B+D=0$, $C+E=0$ and $A=1$. Solve it. $A=1, B=-1, C=0, D=-1, E=0$
So $\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$.

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\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} = \int \frac{1}{x} + \frac{-x}{x^2+1}\Big|_{u=x^2+1,du=2xdx} + \\
\frac{-x}{(x^2+1)^2}\Big|_{u=x^2+1,du=2xdx} dx \\
= \ln|x| - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \\
= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C
\end{aligned} \tag{8}
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