Outline

- Review

Vector is represented by a directed line segment.
Ex: $\vec{AB}$ with $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

$$\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > \tag{1}$$

$$\vec{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k} \tag{2}$$

$$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{3}$$

Two vectors are equal if they have the same length and direction. If $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ are equal, then

$$u_1 = v_1, \quad u_2 = v_2, \quad u_3 = v_3 \tag{4}$$

Vector algebra operations: $\vec{u} =< u_1, u_2, u_3 >$,$\vec{v} =< v_1, v_2, v_3 >$, k is a scalar:

- $\vec{u} \pm \vec{v} =< u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 >$
- $k\vec{u} =< ku_1, ku_2, ku_3 >$

For a vector $\vec{u}$, it consists:

- $|\vec{u}|$ is the length
- $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$

Ex: Find $|\vec{a}$, $\vec{a} + \vec{b}$,$\vec{a} - \vec{b}$,$2\vec{a}$, $3\vec{a} - 4\vec{b}$. Also find a unit vector that has the same direction as $\vec{a}$. $\vec{a} =< 2, 1, -2 >$,$\vec{b} =< 1, 1, 0 >$.

$$|\vec{a}| = 3, \quad \vec{a}+\vec{b} =< 3, 2, -1 >, \quad \vec{a}-\vec{b} =< 1, 0, -2 >, \quad 2\vec{a} =< 4, 2, -4 > \tag{5}$$

$$3\vec{a} - 4\vec{b} =< 2, -1, -6 >, \quad \frac{\vec{a}}{|\vec{a}|} =< \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} > \tag{6}$$

The dot product of $\vec{u} =< u_1, u_2, u_3 >$,$\vec{v} =< v_1, v_2, v_3 >$.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{7}$$

Angle between two vectors:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \tag{8}$$

If the two vectors are orthogonal, then $\vec{u} \cdot \vec{v} = 0$.
The vector projection of $\vec{u}$ onto $\vec{v}$.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} \tag{9}$$

The cross product

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta\vec{n} \qquad (10)$$

If $\vec{u}$ and $\vec{v}$ are not parallel, they determine a plane. $\vec{n}$ is perpendicular to the plane by the right hand rule.
Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad (11)$$

$|\vec{u} \times \vec{v}|$ is the area of a parallelogram.

Ex: Find $\vec{a}\cdot\vec{b}$,$\vec{a}\times\vec{b}$. Verify that $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$. Also find the angle between $\vec{a}$ and $\vec{b}$ $\vec{a} =< 2, -2, 4 >$, $\vec{b} =< -1, 1, 2 >$

$$\vec{a}\cdot\vec{b} = 4 \tag{12}$$

$$\vec{a}\times\vec{b} =< -8, -8, 0 > \tag{13}$$

$$\vec{a}\times\vec{b}\cdot\vec{a} = 0 \tag{14}$$

$$\vec{a}\times\vec{b}\cdot\vec{b} = 0 \tag{15}$$

$$\cos\theta = \frac{1}{3} \tag{16}$$

Ex: Find $\text{proj}_{\vec{v}}\vec{u}$ with $\vec{u} =< 1, 1, -5 >$, $\vec{b} =< 2, 1, -1 >$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{8}{3}\vec{i} + \frac{4}{3}\vec{j} - \frac{4}{3}\vec{i} \tag{17}$$

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$$x = x_0 + tv_1 \quad y = y_0 + tv_2 \quad z = z_0 + tv_3 \quad -\infty < t < \infty \tag{18}$$

An equation for a plane in space:
The plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ has

- vector equation $\vec{n} \cdot \vec{P_0P} = 0$
- component equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
- simplified component equation
  $Ax + By + Cz = D = Ax_0 + By_0 + Cz_0$

The distance from a point S to a line in space through P parallel to $\vec{v}$.

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} \tag{19}$$

The distance from a point to a plane.
If P is a point on a plane with $\vec{n}$, then the distance from a point S to a plane is

$$d = |\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|}| \tag{20}$$

Ex: Find the parametric equations for the line passing through the point $(-2, 4, 10)$ and perpendicular to the plane $3x + y - 8z - 5 = 0$.

$$x = -2 + 3t, \quad y = 4 + t, \quad z = 10 - 8t \tag{21}$$

Ex: Find an equation for the plane passing through three points $P(0, 1, 1)$, $Q(1, 1, 0)$ and $R(1, 0, 1)$.

$$\vec{PQ} =< 1, 0, -1 > \tag{22}$$

$$\vec{PQ} =< 1, -1, 0 > \tag{23}$$

$$\vec{PQ} \times \vec{PR} =< -1, -1, -1 > \tag{24}$$

$$-x - y - z = -2 \tag{25}$$

Ex: Find all first and second order derivatives for the function $f(x, y) = \ln(3x + y)$.

$$f_x = \frac{3}{3x + y^2} \tag{26}$$

$$f_y = \frac{2y}{3x + y^2} \tag{27}$$

$$f_{xx} = \frac{-9}{(3x + y^2)^2} \tag{28}$$

$$f_{xy} = \frac{-6y}{(3x + y^2)^2} \tag{29}$$

$$f_{yx} = \frac{-6y}{(3x + y^2)^2} \tag{30}$$

$$f_{yy} = \frac{6x - 2y^2}{(3x + y^2)^2} \tag{31}$$