Outline

- Alternating series
- Absolute convergence test

Alternating series:
A series in which the terms are alternately positive and negative.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots \tag{1}$$

Alternating series test:
The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ converges if all three of the following conditions are satisfied:

- The u_n are all positive.
- $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N
- $u_n \to 0$

Ex:$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}$

- $\frac{1}{n}$ is positive for every n
- $u_n = \frac{1}{n}$, $u_{n+1} = \frac{1}{n+1}$, $u_n \geq u_{n+1}$
- $u_n = \frac{1}{n} \to 0$

Converges.

Ex:$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \tag{2}$$

Converges.

Ex:$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

$$f(x) = \frac{\sqrt{x}+1}{x+1} \quad f' = \frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2} < 0 \tag{3}$$

$f(x)$ is a decreasing function.

$$u_{n+1} < u_n \tag{4}$$

$$\begin{aligned} &\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \lim_{n\to\infty} \frac{\sqrt{n}(1+\frac{1}{\sqrt{n}})}{n(1+\frac{1}{n})} \\ &\lim_{n\to\infty} \frac{(1+\frac{1}{\sqrt{n}})}{\sqrt{n}(1+\frac{1}{n})} = 0 \end{aligned} \tag{5}$$

Converges.

Absolute and conditional convergence

- A series $\sum a_n$ converges absolutely if the corresponding series of absolute values $\sum |a_n|$ converges.
- A series $\sum a_n$ converges but does not converge absolutely converges conditionally.

The absolute convergence test:
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} |(-1)^{n+1} \frac{1}{n^2}| = \frac{1}{n^2} \tag{6}$$

Converges absolutely.

Ex:$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+3}$
This alternating series converges by alternating series test.

$$\sum_{n=1}^{\infty} |(-1)^{n+1} \frac{1}{n+3}| = \sum_{n=1}^{\infty} \frac{1}{n+3} \tag{7}$$

Diverges by limit comparison test with $\frac{1}{n}$.
Converges conditionally.