Outline

- Alternating series
- Absolute convergence test
Alternating series:
A series in which the terms are alternately positive and negative.

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots \quad (1) \]

Alternating series test:
The series \( \sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots \) converges if all three of the following conditions are satisfied:

- The \( u_n \) are all positive.
- \( u_n \geq u_{n+1} \) for all \( n \geq N \) for some integer \( N \)
- \( u_n \to 0 \)

Ex: \( \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n} \)
- \( \frac{1}{n} \) is positive for every \( n \)
- \( u_n = \frac{1}{n}, u_{n+1} = \frac{1}{n+1}, u_n \geq u_{n+1} \)
- \( u_n = \frac{1}{n} \to 0 \)

Converges.
Ex: $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

(2)

Converges.

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n+1}$

$$f(x) = \frac{\sqrt{x} + 1}{x + 1} \quad f' = \frac{1 - x - 2\sqrt{x}}{2\sqrt{x}(x + 1)^2} < 0$$

(3)

$f(x)$ is a decreasing function.

$$u_{n+1} < u_n$$

(4)

$$\lim_{n \to \infty} \frac{\sqrt{n} + 1}{n + 1} = \lim_{n \to \infty} \frac{\sqrt{n}(1 + \frac{1}{\sqrt{n}})}{n(1 + \frac{1}{n})}$$

(5)

$$\lim_{n \to \infty} \frac{(1 + \frac{1}{\sqrt{n}})}{\sqrt{n}(1 + \frac{1}{n})} = 0$$

Converges.
Absolute and conditional convergence

- A series $\sum a_n$ converges absolutely if the corresponding series of absolute values $\sum |a_n|$ converges.
- A series $\sum a_n$ converges but does not converge absolutely converges conditionally.

The absolute convergence test:
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{1}{n^2}$$ (6)

Converges absolutely.
Ex: \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+3} \)
This alternating series converges by alternating series test.

\[ \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n+3} \right| = \sum_{n=1}^{\infty} \frac{1}{n+3} \]  
(7)

Diverges by limit comparison test with \( \frac{1}{n} \).
Converges conditionally.