Outline

- Substitution Rule for Indefinite Integral;
- Substitution Rule for Definite Integral;
- Area Between Curves
Indefinite integral: \( \int f(x)dx \) is the set of all antiderivatives of \( f(x) \).

- \( f(x) \) is integrand;
- \( dx \) means we are doing integration with respect to \( x \);
- it always includes an arbitrary constant, denoted by \( C \);
Remember the following basic formula:

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \]  
\[ \int \frac{1}{x} \, dx = \ln|x| + C \]  
\[ \int \sin x \, dx = -\cos x + C \]  
\[ \int \cos x \, dx = \sin x + C \]  
\[ \int \sec^2 x \, dx = \tan x + C \]  
\[ \int \sec x \tan x \, dx = \sec x + C \]  
\[ \int \csc^2 x \, dx = -\cot x + C \]  
\[ \int \csc x \cot x \, dx = -\csc x + C \]
Continue:

\[ \int e^x \, dx = e^x + C \quad (2a) \]

\[ \int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C \quad (2b) \]

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C \quad (2c) \]
Ex1: \[ \int x^2 \, dx = \frac{x^3}{3} + C \]  

Ex2: \[ \int x^4 + 3x - 9 \, dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + C \]

Note:
- sum and difference rule for integral

Ex3:
\[
\int \left( w + \frac{1}{\sqrt[3]{w}} \right)(4 - w^2) \, dw \\
= \int \left( 4w - w^3 + 4w^{-\frac{1}{3}} - w^{\frac{5}{3}} \right) \, dw \\
= 2w^2 - \frac{1}{4}w^4 + 6w^{\frac{2}{3}} - \frac{3}{8}w^{\frac{8}{3}} + C
\]

Note:
- write radical function in the form of power
- no product rule \( \int f(x) \cdot g(x) \, dx \neq \int f(x) \, dx \cdot \int g(x) \, dx \)
Ex4:

\[
\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} \, dx = \int 4x^7 - 2x + \frac{15}{x} \, dx = \frac{1}{2}x^8 - x^2 + 15\ln|x| + C
\]  

(6)

Note:

- no quotient rule $\int \frac{f(x)}{g(x)} \, dx \neq \frac{\int f(x) \, dx}{\int g(x) \, dx}$
Chain Rule:

\[ F'(u) = f(u) \quad u = g(x) \quad (7) \]

\[ \frac{d}{dx} F(g(x)) = f(g(x))g'(x) \quad (8) \]

Integrate Eq. (8), we can obtain Substitution Rule:

If \( u = g(x) \) is a differentiable function whose range is an interval I and \( f \) is continuous on I, then

\[ \int f(g(x))g'(x)dx = F(u) = \int f(u)du \quad (9) \]
Useful substitution notation:

\[ \int f(g(x))g'(x)\,dx \bigg|_{u=g(x)}; \, du=g'(x)\,dx = \int f(u)\,du \]  

(10)

Note:

- Remind us using substitution \( u = g(x) \)
- Remind us to substitute \( g(x) \) for \( u \) in the final result.
Ex5:

\[
\int \frac{9r^2}{\sqrt{1 - r^3}} dr \bigg|_{u=1-r^3} = -3r^2 dr; \quad \frac{dr}{du} = \frac{-1}{3r^2} du
\]

\[
= \int \frac{9r^2 \cdot du}{\sqrt{u} \cdot (-3r^2)}
\]

\[
= -3 \int u^{-\frac{1}{2}} du
\]

\[
= -6u^{\frac{1}{2}} = -6\sqrt{u}
\]

\[
= -6\sqrt{1 - r^3} + C
\]
Substitution Rule in Definite Integrals:
If \( g'(x) \) is continuous on the interval \([a, b]\) and \( f \) is continuous on the range of \( g \), then

\[
\int_a^b f(g(x))g'(x) \, dx = F(u) = \int_{g(a)}^{g(b)} f(u) \, du
\]  

(12)
Useful substitution notation:

\[ \int_a^b f(g(x))g'(x)dx \Big|_{u=g(x)} = \int_{g(a)}^{g(b)} f(u)du \]

Note:
- Remind us using substitution \( u = g(x) \)
- Remind us to change the limits.
Ex6:

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx \bigg|_{u=x^2+1; du=2xdx; x=0,u=1; x=\sqrt{3},u=4}$$

$$= \int_1^4 \frac{2}{\sqrt{u}} \, du$$

$$= 4\sqrt{u}\bigg|_1^4$$

$$= 4$$
Area Between Curves:
If \( f \) and \( g \) are continuous with \( f(x) \geq g(x) \) throughout \([a, b]\), then the area of the region between the curve \( y = f(x) \) and \( y = g(x) \) from \( a \) to \( b \) is the integral of \((f - g)\) from \( a \) to \( b \)

\[
A = \int_a^b [f(x) - g(x)] \, dx
\] 

(15)
Ex7: Find the area of the region enclosed by the $y = 2x^2$ and $y = x^4 - 2x^2$.

The intersection points are solutions of the equation system

$$y = 2x^2 \quad y = x^4 - 2x^2 \quad (16)$$

They are:

$$x = -2, y = 8; x = 2, y = 8 \quad (17)$$
$y = 2x^2$ is above $y = x^4 - 2x^2$, so $f(x) = 2x^2$ and $g(x) = x^4 - 2x^2$

$$A = \int_{-2}^{2} [2x^2 - (x^4 - 2x^2)] \, dx$$

$$= \int_{-2}^{2} 4x^2 - x^4 \, dx$$

$$= \frac{4x^3}{3} - \frac{x^5}{5} \bigg|_{-2}^{2}$$

$$= \frac{128}{15}$$