

Outline

- Substitution Rule for Indefinite Integral;
- Substitution Rule for Definite Integral;
- Area Between Curves

Indefinite integral: $\int f(x)dx$ is the set of all antiderivatives of $f(x)$.

- $f(x)$ is integrand;
- dx means we are doing integration with respect to x ;
- it always includes an arbitrary constant, denoted by C ;

Remember the following basic formula:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (1a)$$

$$\int \frac{1}{x} dx = \ln|x| + C \quad (1b)$$

$$\int \sin x dx = -\cos x + C \quad (1c)$$

$$\int \cos x dx = \sin x + C \quad (1d)$$

$$\int \sec^2 x dx = \tan x + C \quad (1e)$$

$$\int \sec x \tan x dx = \sec x + C \quad (1f)$$

$$\int \csc^2 x dx = -\cot x + C \quad (1g)$$

$$\int \csc x \cot x dx = -\csc x + C \quad (1h)$$

Continue:

$$\int e^x dx = e^x + C \quad (2a)$$

$$\int \frac{1}{x^2 + 1} dx = \tan^{-1}x + C \quad (2b)$$

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1}x + C \quad (2c)$$

Ex1:

$$\int x^2 dx = \frac{x^3}{3} + C \quad (3)$$

Ex2:

$$\int x^4 + 3x - 9 dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + C \quad (4)$$

Note:

- sum and difference rule for integral

Ex3:

$$\begin{aligned} & \int \left(w + \frac{1}{\sqrt[3]{w}}\right)(4 - w^2) dw \\ &= \int (4w - w^3 + 4w^{-\frac{1}{3}} - w^{\frac{5}{3}}) dw \\ &= 2w^2 - \frac{1}{4}w^4 + 6w^{\frac{2}{3}} - \frac{3}{8}w^{\frac{8}{3}} + C \end{aligned} \quad (5)$$

Note:

- write radical function in the form of power
- no product rule $\int f(x) \cdot g(x) dx \neq \int f(x) dx \cdot \int g(x) dx$

Ex4:

$$\begin{aligned} & \int \frac{4x^{10} - 2x^4 + 15x^2}{x^3} dx \\ &= \int 4x^7 - 2x + \frac{15}{x} dx \\ &= \frac{1}{2}x^8 - x^2 + 15\ln|x| + C \end{aligned} \tag{6}$$

Note:

- no quotient rule $\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x) dx}{\int g(x) dx}$

Chain Rule:

$$F'(u) = f(u) \quad u = g(x) \quad (7)$$

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x) \quad (8)$$

Integrate Eq. (8), we can obtain Substitution Rule:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = F(u) = \int f(u)du \quad (9)$$

Useful substitution notation:

$$\int f(g(x))g'(x)dx|_{u=g(x);du=g'(x)dx} = \int f(u)du \quad (10)$$

Note:

- Remind us using substitution $u = g(x)$
- Remind us to substitute $g(x)$ for u in the final result.

Ex5:

$$\begin{aligned} & \int \frac{9r^2}{\sqrt{1-r^3}} dr \Big|_{u=1-r^3; du=-3r^2 dr; dr=\frac{-1}{3r^2} du} \\ &= \int \frac{9r^2}{\sqrt{u}} \cdot \frac{du}{(-3r^2)} \\ &= -3 \int u^{-\frac{1}{2}} du \\ &= -6u^{\frac{1}{2}} = -6\sqrt{u} \\ &= -6\sqrt{1-r^3} + C \end{aligned} \tag{11}$$

Substitution Rule in Definite Integrals:

If $g'(x)$ is continuous on the interval $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x))g'(x)dx = F(u) = \int_{g(a)}^{g(b)} f(u)du \quad (12)$$

Useful substitution notation:

$$\int_a^b f(g(x))g'(x)dx \Big|_{u=g(x); du=g'(x)dx; x=a, u=g(a); x=b, u=g(b)} = \int_{g(a)}^{g(b)} f(u)du \quad (13)$$

Note:

- Remind us using substitution $u = g(x)$
- Remind us to change the limits.

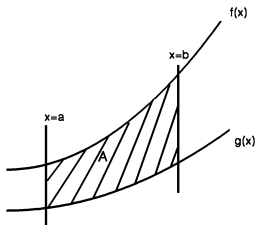
Ex6:

$$\begin{aligned} & \int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx \Big|_{u=x^2+1; du=2x dx; x=0, u=1; x=\sqrt{3}, u=4} \\ &= \int_1^4 \frac{2}{\sqrt{u}} du \\ &= 4\sqrt{u} \Big|_1^4 \\ &= 4 \end{aligned} \tag{14}$$

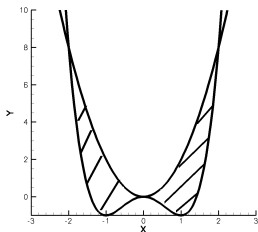
Area Between Curves:

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curve $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b

$$A = \int_a^b [f(x) - g(x)] dx \quad (15)$$



Ex7: Find the area of the region enclosed by the $y = 2x^2$ and $y = x^4 - 2x^2$.



The intersection points are solutions of the equation system

$$y = 2x^2 \quad y = x^4 - 2x^2 \quad (16)$$

They are:

$$x = -2, y = 8; x = 2, y = 8 \quad (17)$$

$y = 2x^2$ is above $y = x^4 - 2x^2$, so $f(x) = 2x^2$ and $g(x) = x^4 - 2x^2$

$$\begin{aligned} A &= \int_{-2}^2 [2x^2 - (x^4 - 2x^2)] dx \\ &= \int_{-2}^2 4x^2 - x^4 dx \\ &= \left. \frac{4x^3}{3} - \frac{x^5}{5} \right|_{-2}^2 \\ &= \frac{128}{15} \end{aligned} \tag{18}$$