Trigonometric Substitutions
Trigonometric substitutions are effective in transforming integrals $\sqrt{a^2 + x^2}, \sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}$ into the integral we can evaluate directly.

$$\sqrt{a^2 + x^2}|_{x=at\tan\theta} = a\sqrt{1 + \tan^2\theta} = a|\sec\theta| \quad (1)$$

After integration, we need to change back to the original variable. So we need the substitution to be reversible. Here, we use $x = at\tan\theta$, we want to be able to set $\theta = \tan^{-1}(\frac{x}{a})$. For reversibility, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \sqrt{a^2 + x^2} = a\sec\theta, \theta = \tan^{-1}(\frac{x}{a}) \quad (2)$$
Ex1:

\[
\int \frac{dx}{\sqrt{4 + x^2}} \bigg|_{x=2\tan\theta} = 2\sec^2\theta d\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2} = \int \frac{2\sec^2\theta}{2\sec\theta} d\theta
\]

\[
= \int \sec\theta d\theta = \int \sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta
\]

\[
= \int \frac{\sec^2\theta + \tan\theta \sec\theta}{\sec\theta + \tan\theta} d\theta |_{u=\sec\theta + \tan\theta} = (\sec^2\theta + \tan\theta \sec\theta) d\theta
\]

\[
= \int \frac{1}{u} du = \ln|u| = \ln|\sec\theta + \tan\theta|
\]

\[
= \ln|\sqrt{4 + x^2} + x| + C
\]

Note: When we write the answer in terms of \(x\), we can do this with some right triangle trig. \(\tan\theta = \frac{x}{2}\), so the adjacent is 2, the hypotenuse is \(\sqrt{4 + x^2}\), the opposite is \(x\).
\[
\sqrt{a^2 - x^2} \bigg|_{x=a \sin \theta} = a \sqrt{1 - \sin^2 \theta} = a |\cos \theta|
\]  
(4)

After integration, we need to change back to the original variable. So we need the substitution to be reversible. Here, we use \( x = a \sin \theta \), we want to be able to set \( \theta = \sin^{-1} \left( \frac{x}{a} \right) \). For reversibility, \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\).

\[-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \sqrt{a^2 - x^2} = a \cos \theta, \theta = \sin^{-1} \left( \frac{x}{a} \right)\]  
(5)
Ex2:

\[ \int \frac{x^2}{\sqrt{9-x^2}} \bigg|_{x=3\sin\theta}^{dx=3\cos\theta\,d\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} = \int \frac{9\sin^2\theta \cdot 3\cos\theta}{3\cos\theta} \, d\theta \]

\[ = 9 \int \sin^2\theta \, d\theta = 9 \int \frac{1 - \cos 2\theta}{2} \, d\theta \]

\[ = \frac{9}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) \]

\[ = \frac{9}{2} \left( \theta - \sin \theta \cos \theta \right) \]

\[ = \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) - \frac{x}{2} \sqrt{9-x^2} + C \]

Note: When we write the answer in terms of \( x \), we can do this with some right triangle trig. \( \sin \theta = \frac{x}{3} \), so the opposite is \( x \), the hypotenuse is 3, the adjacent is \( \sqrt{9-x^2} \).
\[ \sqrt{x^2 - a^2}\big|_{x=a\sec\theta} = a\sqrt{\sec^2\theta - 1} = a|\tan\theta| \]  

(7)

After integration, we need to change back to the original variable. So we need the substitution to be reversible. Here, we use \( x = a\sec\theta \), we want to be able to set \( \theta = \sec^{-1}\left(\frac{x}{a}\right) \). For reversibility, \( 0 \leq \theta < \frac{\pi}{2} \) for \( \frac{x}{a} \geq 1 \) or \( \frac{\pi}{2} < \theta \leq \pi \) for \( \frac{x}{a} \leq -1 \). To simplify calculation, we will restrict its use to integrals in which \( \frac{x}{a} \geq 1 \). So

\[ 0 \leq \theta < \frac{\pi}{2}, \quad \sqrt{x^2 - a^2} = a\tan\theta, \theta = \sec^{-1}\left(\frac{x}{a}\right) \]  

(8)
Ex3:

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - \frac{4}{25}}}$$

$$= \int \frac{\frac{2}{5}\sec\theta\tan\theta d\theta}{5 \cdot \frac{2}{5}\tan\theta}$$

$$= \frac{1}{5} \int \sec\theta d\theta = \frac{1}{5} \ln|\sec\theta + \tan\theta|$$

$$= \frac{1}{5} \ln|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}| + C$$

Note: When we write the answer in terms of $x$, we can do this with some right triangle trig. $\sec\theta = \frac{5x}{2}$, so the hypotenuse is $5x$, the adjacent is 2, and the opposite is $\sqrt{25x^2 - 4}$