Nonhomogeneous ODE:
nth order linear nonhomogeneous ODE:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

Associated homogeneous equation:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{2}$$

For Eq. (2), the general solution is:

$$y = c_1y_1 + c_2y_2 + ... + c_ny_n \tag{3}$$

It is called the complementary function of Eq.(1)
Any solution free of arbitrary parameters y_p satisfying Eq. (1) is called a particular solution.

Thm. 4.6:
The general solution of Eq. (1) is $y = y_c + y_p$

Example 1:

$$y''' - 6y'' + 11y' - 6y = 3x \tag{4}$$

Associated H. E. :

$$y''' - 6y'' + 11y' - 6y = 0 \tag{5}$$

$$m^3 - 6m^2 + 11m - 6 = 0 \tag{6}$$

$$m_{1,2,3} = 1, 2, 3 \tag{7}$$

$$y_c = c_1e^x + c_2e^{2x} + c_3e^{3x} \tag{8}$$

$$y_p = -\frac{11}{12} - \frac{1}{2}x \tag{9}$$

General solution:

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x \tag{10}$$

Thm. 4.7. Superposition Principle:
If y_{pi} is a particular solution of:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x) \tag{11}$$

Then $y_p = y_{p1} + y_{p2} + ... + y_{pk}$ is a particular solution of:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g_1(x) + g_2(x) + ... + g_k(x) \tag{12}$$

Example 2:

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x \tag{13}$$
$$y_{p1} = -4x^2 \tag{14}$$
$$y_{p2} = e^{2x} \tag{15}$$
$$y_{p3} = xe^x \tag{16}$$
$$y_p = -4x^2 + e^{2x} + xe^x \tag{17}$$

Find Particular Solution: method of undetermined coefficients
Conjecture about the form of y_p.
Limitations:

- Coefficients a_i are constants
- $g(x)$ is a constant, a polynomial function, an exponential function $e^{\alpha x}$, a sin or cos function $\sin(\beta x)$ or $\cos(\beta x)$, or finite sums and products of these functions.

Example 3:

$$y'' + 4y' - 2y = 2x^2 - 3x + 6 \tag{18}$$
$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(2+\sqrt{6})x} \tag{19}$$
$$y_p = Ax^2 + Bx + C \tag{20}$$
$$y_p' = 2Ax + B \tag{21}$$
$$y_p'' = 2A \tag{22}$$
$$y'' + 4y' - 2y = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6 \tag{23}$$
$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6 \tag{24}$$
$$A = -1, \quad B = -\frac{5}{2}, \quad C = -9 \tag{25}$$
$$y_p = -x^2 - \frac{5}{2}x - 9 \tag{26}$$
$$y = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9 \tag{27}$$

Example 4:

$$y'' - y' + y = 2\sin(3x) \tag{28}$$

$$\lambda^2 - \lambda + 1 = 0 \tag{29}$$

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \tag{30}$$

$$y_c = e^{\frac{1}{2}x}(c_1 \cos(\frac{\sqrt{3}}{2}x) + c_2 \sin(\frac{\sqrt{3}}{2}x)) \tag{31}$$

$$y_p = A\cos(3x) + B\sin(3x) \tag{32}$$

$$y_p' = -3A\sin(3x) + 3B\cos(3x) \tag{33}$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x) \tag{34}$$

$$y'' - y' + y = (-9A - 3B + A)\cos(3x) + (-9B + 3A + B)\sin(3x) = 2\sin(3x) \tag{35}$$

$$-8A - 3B = 0, \quad 3A - 8B = 2 \tag{36}$$

$$A = \frac{6}{73}, \quad B = -\frac{16}{73} \tag{37}$$

$$y_p = \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x) \tag{38}$$

$$y = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(2+\sqrt{6})x} + \frac{6}{73}\cos(3x) - \frac{16}{73}\sin(3x) \tag{39}$$

Example 5:

$$y'' - 4y' + 3y = 2e^{2x} \tag{40}$$

$$y_c = c_1 e^x + c_2 e^{3x} \tag{41}$$

$$y_p = Ae^{2x} \tag{42}$$

$$y_p' = 2Ae^{2x} \tag{43}$$

$$y_p'' = 4Ae^{2x} \tag{44}$$

$$y'' - 4y' + 3y = (4A - 8A + 3A)e^{2x} = 2e^{2x} \tag{45}$$

$$A = -2 \tag{46}$$

$$y = c_1 e^x + c_2 e^{3x} - 2e^{2x} \tag{47}$$

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