Power series solution(cont.):

Example 1:

$$(x^2+1)y''+xy'-y=0 \tag{1}$$

There has singular points $x=\pm i$. So from Theorem, two power series solutions centerd at 0 exists and are convergent for $|x|<1$.
Assume $y=\sum_{n=0}^{\infty} c_n x^n$

$$y'=\sum_{n=1}^{\infty} nc_n x^{n-1} \tag{2}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \tag{3}$$

$$(x^2+1)y''+xy'-y=(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+x\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} c_n x^n \tag{4}$$

$$=\sum_{n=2}^{\infty} n(n-1)c_n x^n+\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty} nc_n x^n-\sum_{n=0}^{\infty} c_n x^n \tag{5}$$

$$=\sum_{n=2}^{\infty} n(n-1)c_n x^n+2c_2+6c_3x+\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}+c_1x \tag{6}$$

$$+\sum_{n=2}^{\infty} nc_n x^n-c_0-c_1x-\sum_{n=2}^{\infty} c_n x^n \tag{7}$$

$$=2c_2-c_0+6c_3x+\sum_{k=2}^{\infty}[k(k-1)c_k+(k+2)(k+1)c_{k+2}+kc_k-c_k]x^k=0 \tag{8}$$

$$2c_2-c_0=0 \quad 6c_3=0, \quad (k+1)(k-1)c_k+(k+2)(k+1)c_{k+2}=0 \tag{9}$$

$$c_2=\frac{1}{2}c_0, \quad c_3=0 \quad c_{k+2}=\frac{1-k}{k+2}c_k, k=2,3,4\ldots \tag{10}$$

$$k=2, \quad c_4=-\frac{1}{4}c_2=-\frac{1}{2\cdot 4}c_0=-\frac{1}{2^2 2!}c_0 \tag{11}$$

$$k=3, \quad c_5=-\frac{2}{5}c_3=0 \tag{12}$$

$$k=4, \quad c_6=-\frac{3}{6}c_4=\frac{3}{2\cdot 4\cdot 6}c_0=\frac{1\cdot 3}{2^3 3!}c_0 \tag{13}$$

$$k=5, \quad c_7=-\frac{4}{7}=0 \tag{14}$$

$$k=6, \quad c_8=-\frac{5}{8}c_6=-\frac{3\cdot 5}{8}c_6=-\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}c_0=-\frac{1\cdot 3\cdot 5}{2^4 4!}c_0 \tag{15}$$

$$k=7, \quad c_9=-\frac{6}{9}c_7=0 \tag{16}$$

$$k=8, \quad c_{10}=-\frac{7}{10}c_8=\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}c_0=\frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}c_0 \tag{17}$$

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+c_8x^8+c_9x^9+c_{10}x^{10}\ldots \tag{18}$$

$$y=c_0+c_1x+\frac{1}{2}c_0x^2+0-\frac{1}{2^2 2!}c_0x^4+0+\frac{1\cdot 3}{2^3 3!}c_0x^6+0-\frac{1\cdot 3\cdot 5}{2^4 4!}c_0x^8+0 \tag{19}$$

$$+\frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}c_0 x^{10}\dots \tag{20}$$

$$y = c_0 y_1(x) + c_1 y_2(x) \tag{21}$$

$$y_1(x) = 1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^6 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^8 \tag{22}$$

$$+\frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^{10} + \dots \tag{23}$$

$$y_1(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\dots(2n-3)}{2^n n!}x^{2n}, \quad |x| < 1 \tag{24}$$

$$y_2(x) = x \tag{25}$$

Example 2:

$$y'' - (1+x)y = 0 \tag{26}$$

$$c_2 = \frac{1}{2}c_0, \quad c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}, k = 1, 2, 3\dots \tag{27}$$

$$c_1 = 0, \quad c_2 = \frac{1}{2}c_0, \quad c_3 = \frac{1}{6}c_0, \quad c_4 = \frac{1}{24}c_0 \quad c_5 = \frac{1}{30}c_0\dots \tag{28}$$

$$c_0 = 0, \quad c_2 = 0, \quad c_3 = \frac{1}{6}c_1, \quad c_4 = \frac{1}{12}c_1 \quad c_5 = \frac{1}{120}c_1\dots \tag{29}$$

$$y = c_0 y_1(x) + c_1 y_2(x) \tag{30}$$

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \dots \tag{31}$$

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \dots \tag{32}$$

Example 3:

$$y'' + (\cos x)y = 0 \tag{33}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \tag{34}$$

$$y'' + (\cos x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!})\sum_{n=0}^{\infty} c_n x^n \tag{35}$$

$$= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots + (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!})(c_0 + c_1 x + c_2 x^2 + \dots) \tag{36}$$

$$= 2c_2 + c_0 + (6c_3 + c_1)x + (12c_4 + c_2 - \frac{1}{2}c_0)x^2 + (20c_5 + c_3 - \frac{1}{2}c_1)x^3 + \dots) = 0 \tag{37}$$

$$2c_2 + c_0 = 0 \quad 6c_3 + c_1 = 0 \quad 12c_4 + c_2 - \frac{1}{2}c_0 = 0 \quad 20c_5 + c_3 - \frac{1}{2}c_1 = 0 \tag{38}$$

$$c_2 = -\frac{1}{2}c_0 \quad c_3 = -\frac{1}{6}c_1 \quad c_2 = \frac{1}{12}c_0 \quad c_5 = \frac{1}{30}c_1 \tag{39}$$

$$y = c_0 y_1(x) + c_1 y_2(x) \tag{40}$$

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