Solution about singular point (cont.):
Indicial equation:

$$p(x) = xP(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

$$q(x) = x^2 Q(x) = \sum_{n=0}^{\infty} b_n x^n \tag{2}$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{3}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \tag{4}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \tag{5}$$

$$x^2 y'' + x[xP(x)]y' + [x^2 Q(x)]y = 0 \tag{6}$$

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} c_n x^{n+r} \tag{7}$$

$$= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^n + \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} c_n (n+r) x^n + \sum_{n=0}^{\infty} b_n x^n \sum_{n=0}^{\infty} c_n x^n = 0 \tag{8}$$

$$r(r-1) + a_0 r + b_0 = 0 \tag{9}$$

Example 1:

$$3xy'' + y' - y = 0 \tag{10}$$

$$p(x) = xP(x) = x\frac{1}{3x} = \frac{1}{3} \tag{11}$$

$$q(x) = x^2 Q(x) = -x^2 \frac{1}{3x} = -\frac{1}{3}x \tag{12}$$

$$a_0 = \frac{1}{3} \quad b_0 = 0 \tag{13}$$

$$r(r-1) + \frac{1}{3}r = 0 \tag{14}$$

$$r = 0 \quad r = \frac{2}{3} \tag{15}$$

Case I: $|r_1 - r_2|$ is not an integer:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \tag{16}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \tag{17}$$

Case II:$|r_1 - r_2|$ is an integer $(r_1 > r_2)$:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \tag{18}$$

$$y_2(x) = C y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \tag{19}$$

Case III:$r_1 = r_2$:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \tag{20}$$

$$y_2(x) = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \tag{21}$$

Example 2:

$$2xy'' + (1+x)y' + y = 0 \tag{22}$$

$x = 0$ is a regular singular point for the equation. We try to find the solution in the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$2xy'' + (1+x)y' + y \tag{23}$$

$$= x^r[r(2r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k] \tag{24}$$

$$r(2r-1) = 0 \tag{25}$$

$$(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k = 0 \tag{26}$$

$$r = \frac{1}{2}, \quad c_{k+1} = \frac{-c_k}{2(k+1)}, k = 0, 1, \dots \tag{27}$$

$$r = 0, \quad c_{k+1} = \frac{-c_k}{2k+1}, k = 0, 1, \dots \tag{28}$$

From $r = \frac{1}{2}$, we find:

$$c_n = \frac{(-1)^n c_0}{2^n n!} \tag{29}$$

From $r = 0$, we find:

$$c_n = \frac{(-1)^n c_0}{1 \cdot 3 \cdot 5 \dots (2n-1)} \tag{30}$$

$$y_1(x) = x^{\frac{1}{2}}[1 + \sum_{k=1}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+\frac{1}{2}}] \tag{31}$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \dots (2n-1)} x^n \tag{32}$$

$$y = c_0 y_1(x) + c_1 y_2(x) \tag{33}$$

Example 3:

$$xy'' + y = 0 \tag{34}$$

$$xP(x) = 0, \quad x^2Q(x) = x \tag{35}$$

$$r(r-1) = 0 \tag{36}$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{37}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \tag{38}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \tag{39}$$

$$xy'' + y = \tag{40}$$

$$x^r[\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n] = 0 \tag{41}$$

$$r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+1+r)(k+r)c_{k+1} + c_k]x^k = 0 \tag{42}$$

$$r = 1 \quad c_{k+1} = -\frac{c_k}{(k+2)(k+1)} \quad k = 0, 1, 2 \ldots \tag{43}$$

$$r = 0 \quad c_{k+1} = -\frac{c_k}{(k+1)k} \quad k = 1, 2 \ldots \tag{44}$$

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} \tag{45}$$

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} \tag{46}$$

So $y_2(x)$ is the same as $y_1(x)$, we have to use reduction of order to find out $y_2(x)$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx \tag{47}$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int 0dx}}{y_1^2(x)} dx \tag{48}$$

$$y_2(x) = y_1(x) \int \frac{dx}{[x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \ldots]^2} \tag{49}$$

$$y_2(x) = y_1(x) \int \frac{dx}{[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{12}x^5 + \ldots]} \tag{50}$$

$$y_2(x) = y_1(x) \int [\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19x}{72} + \ldots]dx \tag{51}$$

$$y_2(x) = y_1(x)[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \ldots] \tag{52}$$

$$y_2(x) = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \tag{53}$$

Hw: Page 257, No. 13, 14, 23