

Laplace Transform:

Definition: Laplace transform:

Let f be a function defined for $t \geq 0$. The integral

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (1)$$

is Laplace transform of f if this integral converges.

Example 1: Evaluate $L\{1\}$.

$$\begin{aligned} L\{1\} &= \int_0^{\infty} e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. \frac{-e^{-st}}{s} \right|_0^b = \lim_{b \rightarrow \infty} \frac{-e^{-sb} + 1}{s} = \frac{1}{s} \quad s > 0 \end{aligned} \quad (2)$$

Example 2: Evaluate $L\{t\}$.

$$L\{t\} = \int_0^{\infty} e^{-st} t dt = \left. \frac{-te^{-st}}{s} \right|_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt = \frac{1}{s} L\{1\} = \frac{1}{s^2} \quad s > 0 \quad (3)$$

Example 3: Evaluate $L\{e^{-3t}\}$.

$$L\{e^{-3t}\} = \int_0^{\infty} e^{-st} e^{-3t} dt = \int_0^{\infty} e^{-(s+3)t} dt = \left. \frac{-e^{-(s+3)t}}{(s+3)} \right|_0^{\infty} = \frac{1}{s+3} \quad s > -3 \quad (4)$$

Example 4: Evaluate $L\{\sin(2t)\}$.

$$\begin{aligned} L\{\sin(2t)\} &= \int_0^{\infty} e^{-st} \sin(2t) dt = \left. \frac{-e^{-st} \sin(2t)}{s} \right|_0^{\infty} + \frac{2}{s} \int_0^{\infty} e^{-st} \cos(2t) dt \\ &= \frac{2}{s} \int_0^{\infty} e^{-st} \cos(2t) dt \quad s > 0 \\ &= \frac{2}{s} \left[\left. \frac{-e^{-st} \cos(2t)}{s} \right|_0^{\infty} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin(2t) dt \right] \\ &= \frac{2}{s^2} - \frac{4}{s^2} L\{\sin 2t\} \end{aligned} \quad (5)$$

$$L\{\sin(2t)\} = \frac{2}{s^2 + 4} \quad (6)$$

Piecewise continuous::

For a function f , in any interval $a \leq t \leq b$, there are at most a finite number of points, at which f has finite discontinuity, and is continuous on each open interval.

Example 5: Evaluate:

$$f(t) = \begin{cases} 0 & 0 \leq t < 3 \\ 2 & t \geq 3 \end{cases} \quad (7)$$

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^3 e^{-st} (0) dt + \int_3^{\infty} e^{-st} (2) dt \\ &= 0 + \left. \frac{2e^{-st}}{-s} \right|_3^{\infty} \\ &= \frac{2e^{-3s}}{s}, \quad s > 0 \end{aligned} \quad (8)$$

Transforms of some basic functions:

- $L\{1\} = \frac{1}{s}$
- $L\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3 \dots$
- $L\{e^{at}\} = \frac{1}{s-a}$
- $L\{\sin kt\} = \frac{k}{s^2+k^2}$
- $L\{\cos kt\} = \frac{s}{s^2+k^2}$
- $L\{\sinh kt\} = \frac{k}{s^2-k^2}$
- $L\{\cosh kt\} = \frac{s}{s^2-k^2}$

Property: L is a linear transform:

Linear combination:

$$\int_0^{\infty} e^{-st} [\alpha f(t) + \beta g(t)] dt = \alpha L\{f(t)\} + \beta L\{g(t)\} = \alpha F(s) + \beta G(s) \quad (9)$$

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