Operational Properties
Theorem: Second Translation Theorem:
If $F(s) = L\{f(t)\}$ and $a > 0$ then

$$L\{f(t-a)U(t-a)\} = e^{-as}F(s) \tag{1}$$

Proof:

$$L\{f(t-a)U(t-a)\} = \int_0^\infty e^{-st}f(t-a)U(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt \tag{2}$$

$$u = t - a \tag{3}$$

$$L\{f(t-a)U(t-a)\} = \int_0^\infty e^{-s(u+a)}f(u)du = e^{-sa}\int_0^\infty e^{-su}f(u)du = e^{-sa}F(s) \tag{4}$$

Inverse form of second translation theorem:

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a) \tag{5}$$

Example 1: Evaluate $L^{-1}\{\frac{1}{s-4}e^{-2s}\}$

$$L^{-1}\{\frac{1}{s-4}e^{-2s}\} = e^{4(t-2)}U(t-2) \tag{6}$$

Example 2: Evaluate $L^{-1}\{\frac{s}{s^2+9}e^{\frac{-\pi s}{2}}\}$

$$L^{-1}\{\frac{s}{s^2+9}e^{\frac{-\pi s}{2}}\} = \cos 3(t-\frac{\pi}{2})U(t-\frac{\pi}{2}) \tag{7}$$

Alternative form of second translation theorem:

$$L\{g(t)U(t-a)\} = e^{-as}L\{g(s+a)\} \tag{8}$$

Proof:

$$L\{g(t)U(t-a)\} = \int_0^\infty e^{-st}g(t)U(t-a)dt = \int_a^\infty e^{-st}g(t)dt \tag{9}$$

$$u = t - a \tag{10}$$

$$\begin{aligned} L\{g(t)U(t-a)\} &= \int_0^\infty e^{-s(u+a)}g(u+a)du = e^{-sa}\int_0^\infty e^{-su}g(u+a)du \\ &= e^{-sa}L\{g(s+a)\} \end{aligned} \tag{11}$$

Example 3: Evaluate $L\{tU(t-2)\}$

$$g(t) = t \tag{12}$$

$$L\{g(t+a)\} = L\{t+2\} = \frac{1}{s^2} + \frac{2}{s} \tag{13}$$

$$L\{tU(t-2)\} = e^{-2s}(\frac{1}{s^2} + \frac{2}{s}) \tag{14}$$

Example 4: Evaluate $L\{\text{cos}tU(t-\pi)\}$

$$g(t) = \cos(t) \tag{15}$$

$$L\{g(t+a)\} = L\{\cos(t+\pi)\} = L\{-\cos(t)\} = -\frac{s}{s^2+1} \tag{16}$$

$$L\{\cos(t)U(t-\pi)\} = e^{-\pi s}(-\frac{s}{s^2+1}) = -\frac{s}{s^2+1}e^{-\pi s} \tag{17}$$

Example 5: Solve $y' + y = f(t), \quad y(0) = 5$, where:

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\text{cos}(t), & t \ge \pi \end{cases} \tag{18}$$

$$f(t) = 3\text{cos}(t)U(t-\pi) \tag{19}$$

$$L\{y'\} + L\{y\} = L\{\text{cos}tU(t-\pi)\} \tag{20}$$

$$sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s} \tag{21}$$

$$Y(s) = \frac{5}{s+1} - 3[-\frac{1}{2}\frac{1}{s+1}e^{-\pi s} + \frac{1}{2}\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{2}\frac{s}{s^2+1}e^{-\pi s}] \tag{22}$$

$$y(t) = 5e^{-t} - \frac{3}{2}[-e^{-(t-\pi)} + \sin(t-\pi) + \cos(t-\pi)]U(t-\pi) \tag{23}$$

$$y(t) = 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \text{sin}t + \text{cos}t]U(t-\pi) \tag{24}$$

$$y(t) = \begin{cases} 5e^{-t} & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\text{sin}t + \frac{3}{2}\text{cos}t & 0 \ge \pi \end{cases} \tag{25}$$

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