Separable variables:
A first-order differential equation is said to be separable or have separable variables if it has a form of $\frac{dy}{dx} = f(x)g(y)$.
Example:

$$\frac{dy}{dx} = y^2 x e^{3x+4y} \tag{1}$$

$$\frac{dy}{dx} = y + \sin x \tag{2}$$

Solving separable differential equation is fairly easy. We first rewrite the differential equation as:

$$h(y)dy = f(x)dx \tag{3}$$

Then you integrate on both sides:

$$\int h(y)dy = \int f(x)dx \tag{4}$$

Note:

- No need to use two constants. A one-parameter family of solutions with implicit form are obtained.
- We will also have to worry about the interval of validity for many of these solutions. Recall that the interval of validity was the range of the independent variable, x in this case, on which solution is valid.
- We need to avoid divison by zero, complex numbers, logarithms of negative numbers or zero, etc. Most of the solutions that we will get from separable differential equations will not be valid for all values of x.

Review integrals:

$$\int \frac{1}{x} dx = \ln|x| + c \tag{5}$$

$$\int \sin x dx = -\cos x + c \tag{6}$$

$$\int x^p dx = \frac{x^{p+1}}{p+1} + c \tag{7}$$

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + c \tag{8}$$

$$\int uv' = uv - \int vu' \tag{9}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c \tag{10}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c, \tag{11}$$

Example 1:

$$(1+x)dy - ydx = 0 \tag{12}$$

$$\frac{dy}{dx} = \frac{y}{1+x} \tag{13}$$

$$\frac{dy}{y} = \frac{dx}{1+x} \tag{14}$$

$$\ln|y| = \ln|1+x| + \ln|c| \tag{15}$$

$$\ln|y| = \ln|c(1+x)| \tag{16}$$

$$y = c(1+x) \tag{17}$$

Example 2:

$$\frac{dy}{dx} = -\frac{x}{y}, y(4) = -3 \tag{18}$$

$$ydy = -xdx \tag{19}$$

$$x^2 + y^2 = c \tag{20}$$

$$x = 4, y = -3, c = 25 \tag{21}$$

$$x^2 + y^2 = 25 \tag{22}$$

Example 3:

$$\frac{dy}{dx} = 6y^2 x, \quad y(1) = \frac{1}{25} \tag{23}$$

$$y^{-2} dy = 6xdx \tag{24}$$

$$-\frac{1}{y} = 3x^2 + c \tag{25}$$

$$x = 1, y = \frac{1}{25}, c = -28 \tag{26}$$

$$y(x) = \frac{1}{28 - 3x^2} \tag{27}$$

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, x=1 in this case.
So, for this case, we've got avoid two values of x. Namely, $x \neq \sqrt{\frac{28}{3}}$,and $x \neq -\sqrt{\frac{28}{3}}$. This gives us three possible intervals of validity.

Example4:

$$\frac{dy}{dx} = (y^2 - 4) \tag{28}$$

$$\frac{dy}{y^2 - 4} = dx \tag{29}$$

$$\frac{1}{4}(\frac{1}{y-2} - \frac{1}{y+2})dy = dx \tag{30}$$

$$\frac{1}{4}\ln|\frac{y-2}{y+2}| = x + c \tag{31}$$

$$\frac{y-2}{y+2} = ce^{4x} \tag{32}$$

or:

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}} \tag{33}$$

Lose solution:

$$y = -2 \tag{34}$$

Example5:

$$e^y \sin 2x dx = (e^{2y} - y)\cos x dy, y(0) = 0 \tag{35}$$

$$\frac{e^{2y} - y}{e^y} dy = \frac{\sin 2x}{\cos x} dx \tag{36}$$

$$(e^y - ye^{-y})dy = 2\sin x dx \tag{37}$$

$$e^y + ye^{-y} + e^{-y} = -2\cos x + c \tag{38}$$

$$x = 0, y = 0, c = 4 \tag{39}$$

$$e^y + ye^{-y} + e^{-y} = -2\cos x + 4 \tag{40}$$

Finding intervals of validity from implicit solutions can often be very difficult so we will also not bother with that for this problem.

Hw: Page 54: No.5,7,9,14,24,27.