Exact differential equations:

Example:

$$(2x-5y)dx+(-5x+3y^2)dy=0 \tag{1}$$

Let's start off by supposing that there is a function $f(x,y)$, for this example it is $x^2-5xy+y^3$. Take some partial derivative of the function:
$f_x=2x-5y$
$f_y=-5x+3y^2$
Now compare these partial derivatives to the differential equation and notice that with these, we can now write the differential equations as

$$f_x dx+f_y dy=0 \tag{2}$$

Recall Differential of a function of two variables:
If $f=f(x,y)$ is a function of two variables with continuous first partial derivatives, then $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$.
If $f(x,y)=c$, then $\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$.
So we have:

$$x^2-5xy+y^3=c \tag{3}$$

This is an implicit one parameter family of solution.

So if a first order DE is written in differential form $M(x,y)dx+N(x,y)dy=0$, and suppose there is a function which can be written in $f(x,y)$. If $f_x=M(x,y)$ and $f_y=N(x,y)$, then we call the differential equation is exact. If it is, we can rewrite it as $\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$. So the solution is $f(x,y)=c$

Finding the function is the central task in an exact differential equation. Before we do it, it is useful to find out whether the differential equation is exact.

Thm: DE written in differential form $M(x,y)dx+N(x,y)dy=0$. Suppose M, N and their first partial derivative are continuous in $R: a<x<b, c<y<d$. DE is exact if and only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof:
Necessity: If $M(x,y)dx+N(x,y)dy=0$ is exact, then there exists f that $M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$.
Therefore:
$M(x,y)=\frac{\partial f}{\partial x}, N(x,y)=\frac{\partial f}{\partial y}$,
and $\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=\frac{\partial N}{\partial x}$

Sufficiency:Suppose $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Now construct $z=f(x,y)$ such that $df=Mdx+Ndy$.

First assume $\frac{\partial f}{\partial x} = M$, then
$f = \int M(x,y)dx + g(y)$. Then
$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M(x,y)dx + g'(y)$.
But $\frac{\partial f}{\partial y} = N$.
So $g'(y) = N - \frac{\partial}{\partial y} \int M(x,y)dx$.
Integrate to get g, and put it back to get f.
Now check for $z = f(x,y)$., whether $dz = Mdx + Ndy$.
We find that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M + g' = N$.

Example 1:

$$2xydx + (x^2 - 1)dy = 0 \tag{4}$$

$$M = 2xy \quad N = x^2 - 1 \tag{5}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x \tag{6}$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{7}$$

$$\frac{\partial f}{\partial x} = M = 2xy \tag{8}$$

$$f = \int Mdx + g(y) = x^2y + g(y) \tag{9}$$

$$\frac{\partial f}{\partial y} = N \tag{10}$$

$$x^2 + g'(y) = N = x^2 - 1 \tag{11}$$

$$g'(y) = -1 \tag{12}$$

$$g(y) = -y \tag{13}$$

$$f = x^2y - y \tag{14}$$

Solution:

$$x^2y - y = c \tag{15}$$

Example 2:

$$(e^{2y} - y\cos(xy))dx + (2xe^{2y} - x\cos(xy) + 2y)dy = 0 \tag{16}$$

$$M = e^{2y} - y\cos(xy) \quad N = 2xe^{2y} - x\cos(xy) + 2y \tag{17}$$

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin(xy) - \cos xy = \frac{\partial N}{\partial x} \tag{18}$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{19}$$

$$\frac{\partial f}{\partial y} = N = 2xe^{2y} - x\cos(xy) + 2y \tag{20}$$

$$f = \int N dy + h(x) = xe^{2y} - \sin(xy) + y^2 + h(x) \tag{21}$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos(xy) + h'(x) = e^{2y} - y\cos(xy) \tag{22}$$

$$h'(x) = 0 \tag{23}$$

$$h(x) = c \tag{24}$$

$$f = xe^{2y} - \sin(xy) + y^2 = c \tag{25}$$

Example 3:

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2 \tag{26}$$

$$(\cos x \sin x - xy^2)dx + y(1-x^2)dy = 0 \tag{27}$$

$$M = \cos x \sin x - xy^2 \quad N = y(1-x^2) \tag{28}$$

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x} \tag{29}$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{30}$$

$$\frac{\partial f}{\partial y} = y(1-x^2) \tag{31}$$

$$f = \int N dy + h(x) = \frac{y^2}{2}(1-x^2) + h(x) \tag{32}$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2 \tag{33}$$

$$h'(x) = \cos x \sin x \tag{34}$$

$$h(x) = -\frac{1}{2}\cos^2 x \tag{35}$$

$$\frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x = c \tag{36}$$

$$x = 0 \quad y = 2 \quad c = 6 \tag{37}$$

$$y^2(1-x^2) - \cos^2 x = 3 \tag{38}$$

Integral factor μ

$$\mu M(x,y)dx + \mu N(x,y)dy = 0 \tag{39}$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \tag{40}$$

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N \tag{41}$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu \tag{42}$$

Special form $\mu = \mu(x)$

$$\mu_x N = (M_y - N_x)\mu \tag{43}$$

$$\frac{d\mu}{\mu} = (\frac{M_y - N_x}{N})dx \tag{44}$$

Hope: $\frac{M_y - N_x}{N}$ depend on x, solve for $\mu(x)$
Similarly, if $(\frac{N_x - M_y}{M})$ depend only on y,solve for $\mu(y)$

Example 4:

$$xydx + (2x^2 + 3y^2 - 20)dy = 0 \tag{45}$$

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2 + 3y^2 - 20} \tag{46}$$

$$\frac{N_x - M_y}{M} = \frac{3}{y} \tag{47}$$

$$\frac{d\mu}{\mu} = \frac{3}{y}dy \tag{48}$$

$$\mu = y^3 \tag{49}$$

$$xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0 \tag{50}$$

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c \tag{51}$$

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