You have 75 minutes (the full class period) to answer the following 7 (seven) questions on 3 (three) pages. Show all your work. There is a total of 210 points on the test (10 of which are extra credit).

1. (10 + 10 + 10 = 30 points)  
   Given the following three vectors in $\mathbb{R}^3$

   \[
   \begin{bmatrix}
   1 \\
   0 \\
   2
   \end{bmatrix},
   \begin{bmatrix}
   0 \\
   1 \\
   0
   \end{bmatrix},
   \begin{bmatrix}
   2 \\
   0 \\
   5
   \end{bmatrix}
   \]

   (a) are these vectors linearly independent?  
   (b) do these vectors span $\mathbb{R}^3$?  
   (c) do these vectors form a basis of $\mathbb{R}^3$?

2. (10 + 10 + 10 = 30 points)  
   Given the following three vectors in $\mathbb{R}^4$

   \[
   \begin{bmatrix}
   1 \\
   0 \\
   2 \\
   0
   \end{bmatrix},
   \begin{bmatrix}
   0 \\
   1 \\
   0 \\
   5
   \end{bmatrix},
   \begin{bmatrix}
   2 \\
   0 \\
   0
   \end{bmatrix}
   \]

   (a) are these vectors linearly independent?  
   (b) do these vectors span $\mathbb{R}^4$?  
   (c) do these vectors form a basis of $\mathbb{R}^4$?
3. (30 points)
Given a transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is linear, a basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^2$, and that the matrix of $T$ with respect to the basis $\beta$ is

$$B = \begin{bmatrix} 1 & 4 \\ 3 & 4 \end{bmatrix}$$

find the standard matrix, $A$, of $T$

4. (25 points)
Given the following basis of $\mathbb{R}^{2\times2}$,

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

find the $\beta$-coordinates of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

5. (15 + 15 + 15 = 45 points)
Given the linear transformation $T$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$, where

$$T(M) = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} M$$

(a) is $T$ linear? (show why or why not)
(b) is $T$ an isomorphism? (show why or why not)
(c) find the matrix of $T$ with respect to the standard basis of $\mathbb{R}^{2\times2}$
6. (10 + 10 + 10 + 10 = 40 points)
Given the matrix
\[
A = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 2 & 0 \\
1 & 0 & -2 \\
-1 & 1 & 1
\end{bmatrix}
\]

(a) find a basis for the kernel of \(A\)
(b) what is the dimension of the kernel of \(A\)?
(c) find a basis for the image of \(A\)
(d) what is the dimension of the image of \(A\)?

7. (10 points – Extra Credit)
Given the matrix \(A\) from above (Problem #6)

\[
\begin{bmatrix}
1 \\
0 \\
2 \\
-1
\end{bmatrix}
\]

(a) Is the vector \(\vec{x}\) in \(\text{im}(A)\)?
(b) If so, what are the coordinates of \(\vec{x}\) with respect to the basis \(\beta\),
where \(\beta\) is the basis of \(\text{im}(A)\) found in part (c) of Problem #6 above?