1. (25 points)
Given that the linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ rotates all vectors clockwise by $45^\circ$ and multiplies all vectors by $\sqrt{2}$, find the matrix $A$ of the linear transformation $T$

2. (5 + 5 + 5 + 5 + 5 + 5 = 30 points)
Given the following system of equations

\[
\begin{align*}
x_1 - 2x_2 + 4x_3 - x_4 &= 9 \\
x_1 - x_2 + 2x_3 - x_4 &= 5 \\
-x_1 + x_2 - 3x_3 + 2x_4 &= -8 \\
x_1 + 2x_3 - 3x_4 &= 7
\end{align*}
\]

(a) Write the system of equations in augmented matrix form
(b) Use elimination to reduce the system to reduced-row echelon form
(c) Is this a consistent system?
(d) How many solutions does this system have?
(e) Write the solution(s) to the system of equations
(f) Is the coefficient matrix invertible?
3. (25 points)
Given the following three matrices

\[ A = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \]

Does the matrix product ABC exist?
If so, what is ABC?

4. (25 points)
Given the following matrix

\[ A = \begin{bmatrix} 1 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix} \]

Is A invertible (be sure to explain why or why not)?
If so, what is the inverse of A?

5. (25 points)
Given the linear transformation T from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), where

\[ T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} \]

find the matrix of T with respect to the basis

\[ \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \]
6. (5 + 5 + 10 + 5 = 25 points)
Given the following matrix

\[
A = \begin{bmatrix}
1 & -2 & 4 & -3 \\
1 & -1 & 2 & -1 \\
-1 & 0 & 0 & -1 \\
1 & 1 & -2 & 3
\end{bmatrix}
\]

(a) Find a basis for the image of \( A \)
(b) Find a basis for the kernel of \( A \)
(c) What is the rank of \( A \)?
(d) What is the nullity of \( A \)?

7. (15 + 15 + 15 = 45 points)
Given the linear transformation \( T \) from \( L^{2 \times 2} \) to \( L^{2 \times 2} \) (lower triangular \( 2 \times 2 \) matrices), where

\[
T(M) = \begin{bmatrix}
1 & 0 \\
3 & 2
\end{bmatrix} M
\]

(a) Find the kernel of \( T \)
(b) Is \( T \) an isomorphism? (show why or why not)
(c) Find the matrix of \( T \) with respect to the basis

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]
8. (30 points)
Find \( \det(A) \), where

\[
A = \begin{bmatrix}
1 & -1 & 4 & -1 \\
-2 & 3 & 1 & 2 \\
0 & 0 & -2 & 0 \\
3 & 2 & -3 & -2
\end{bmatrix}
\]

9. (10 + 10 + 10 = 30 points)
Given the matrix

\[
A = \begin{bmatrix}
1 & -2 & -1 \\
0 & -1 & -1 \\
0 & 6 & 4
\end{bmatrix}
\]

(a) Find all the eigenvalues of \( A \)
(b) Find the eigenvectors associated with each eigenvalue of \( A \)
(c) If possible, find an invertible \( S \) matrix and diagonal \( D \) matrix such that \( D = S^{-1}AS \)

10. (20 + 20 = 40 points)
Given the matrix

\[
A = \begin{bmatrix}
4 & 0 \\
1 & 2
\end{bmatrix}
\]

(a) Find \( A^p \) (where \( p \) is a positive integer)
(b) Find the vector

\[
A^p \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]