Solutions to Selected Problems of Test 2

2-(1) See examples 3.36 and 3.37

2-(2) See exercise 7, 4.3

3. 1-T, 2-T, 3-T, 4-F, 5-F, 6-F, 7-T, 8-F, 9-T, 10-F

4.

(1) Consult our textbook.

(2) Let \( f(x) = e^x - \cos x - 1 \). Note that \( f \) is continuous on \( R \) and \( f(0) = -1 < 0 \) and \( f(\frac{\pi}{2}) > 0 \). Hence by the Intermediate Value Theorem, there exists \( x^* \in (0, \frac{\pi}{2}) \) such that \( f(x^*) = 0 \).

(3) Since \( f \) is continuously differentiable over \([a, b]\), \( f' \) is continuous over \([a, b]\). Hence \( |f'(x)| < M \) for some \( M > 0 \) and for all \( x \in [a, b] \). Hence using MVT, for \( x, y \in [a, b] \), there exists \( c \in (a, b) \) such that \( f(x) - f(y) = f'(c)(x - y) \). Hence, given \( \epsilon > 0 \), choose \( \delta = \epsilon/M \) so that for \( |x - y| < \delta \), we have

\[
|f(x) - f(y)| = |f'(c)(x - y)| \leq M|x - y| < M \cdot \delta = M \cdot \epsilon/M = \epsilon.
\]

(4) \( \Rightarrow \) For this direction, there is nothing to prove, since if \( f \) is continuous over \((0, \infty)\), then it is certainly continuous at 1.

\( \Leftarrow \) Since \( f \) is continuous at 1, given \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) such that \( |1 - a| < \delta_1 \) implies \( |f(1) - f(a)| < \epsilon \). Now for any \( x \in (0, \infty) \), choose \( \delta = x \cdot \delta_1 \), then \( |1 - \frac{y}{x}| < \delta_1 \), or equivalently \( |x - y| < \delta \), we have \( |f(x) - f(y)| = |f(\frac{y}{x})| < \epsilon \).