ECE 601 Midterm Exam 1

Old Dominion University
Fall Semester 2018

Name: __________________________

**Instructions:**

(a) The exam is closed-book. You are allowed only ONE 8.5 x 11 inch sheet of handwritten notes, both sides, and no electronic rendering of any kind.

(b) Use your own calculator. No calculator sharing is allowed during the exam.

(c) No other electronic devices are permitted.

(d) Read each problem statement carefully and show all of your work.

(e) Write your solution to each problem in the space provided below the problem statement.

(f) The exam period is 5:30-7:00 pm (duration: 1.5 hours).

(g) Your signature above indicates that you understand the University Honor System and agree to abide by its terms.

**Score:**

\[
\begin{array}{ccc}
1: & /30 & 3: & /40 \\
2: & /30 & \text{Total:} & /100 \\
\end{array}
\]
1. Matrix Calculations: Suppose a linear mapping is represented by the matrix 

\[ A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}. \]

(a) {10 points} Compute the rank, determinant and trace of \( A \).

(b) {10 points} Compute the condition numbers of \( A \) and \( A^T \).

(c) {10 points} Determine the norm of \( A \).

(a) \( \text{rank}(A) = 1 \), \( \text{det}(A) = 0 \), \( \text{Tr}(A) = 3 \)

(b) Since \( \text{rank}(A) = 1 \), \( \sigma_1(A) > 0 \), \( \sigma_2(A) = 0 \).

\[ \Rightarrow \sigma(A) = \sigma(A^T) = \sigma_1(A)/\sigma_2(A) = +\infty. \]

(c) \( \|A\|_2^2 = \sigma_1^2 = \gamma_{\text{max}}(A^TA) \)

\[ A^TA = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \]

\[ \text{Tr}(A^TA) = 25 = \sigma_1(A) + \sigma_2(A) \quad \Rightarrow \quad \|A\|_2 = 5 \]

Remark: See Homework #3, Problems 1, 3 and Practice Problems Set #1, Problem 3
2. State Space Linearization: A certain electromechanical system has the following nonlinear space model:

\[ \dot{x} = \begin{bmatrix} x_1 x_2 + u \\ x_1^2 + x_2^2 - 1 \end{bmatrix} = F(x_1, u) \]
\[ y = \sin(x_1 u) + \cos(x_2 u) = H(x_1, u) \]

(a) \{15 points\} Describe all the equilibria \((x_e, u_e)\) of the system.

(b) \{15 points\} Derive the linearized state space model for the system about a given \((x_e, u_e)\).

(a) \((x_e, u_e)\) is any solution of

\[ x_{1e} x_{2e} + u_e = 0, \quad x_{1e}^2 + x_{2e}^2 - 1 = 0 \]

(b) \[ A = \frac{\partial F}{\partial x} \bigg|_{(x_e,u_e)} = \begin{bmatrix} x_{2e} & x_{1e} \\ 2x_{1e} & 2x_{2e} \end{bmatrix} \]
\[ b = \frac{\partial F}{\partial u} \bigg|_{(x_e,u_e)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ C = \frac{\partial H}{\partial x} \bigg|_{(x_e,u_e)} = \begin{bmatrix} u_e \cos(x_{1e} u_{1e}) - u_e \sin(x_{2e} u_{2e}) \end{bmatrix} \]
\[ d = \frac{\partial H}{\partial u} \bigg|_{(x_e,u_e)} = x_{1e} \cos(x_{1e} u_{1e}) - x_{2e} \sin(x_{2e} u_{2e}) \]

Remark: See Homework #2, Problems 1, 2.
3. **Operators on** $\mathbb{R}^{n \times n}$: Consider the following maps on the vector space $\mathbb{R}^{n \times n}$.

\[
D : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} : A \mapsto \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})
\]

\[
\bar{D} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} : A \mapsto A - D(A).
\]

For example, if $A$ is the matrix in Problem 1 then

\[
D(A) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \quad \bar{D}(A) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.
\]

(a) \{10 points\} Are $D$ and $\bar{D}$ **linear** operators? Justify your answer explicitly.

(b) \{15 points\} Determine the null space and range space of each operator.

(c) \{15 points\} Determine the eigen-pairs $(\lambda, P)$ of $D$ and $\bar{D}$.

\[(a) \quad D(\alpha A + \beta B) = \text{diag}(\alpha a_{11} + \beta b_{11}, \alpha a_{22} + \beta b_{22})
\]

\[
= \alpha D(A) + \beta D(B) \rightarrow \text{linear}
\]

\[
\bar{D}(\alpha A + \beta B) = \alpha A + \beta B - D(\alpha A + \beta B)
\]

\[
= \alpha (A - D(A)) + \beta (B - D(B))
\]

\[
= \alpha \bar{D}(A) + \beta \bar{D}(B) \rightarrow \text{linear}
\]

\[(b) \quad \text{Null}(D) = \left\{ A \in \mathbb{R}^{n \times n} : a_{ii} = 0 + i \right\}
\]

\[
\text{Range}(D) = \left\{ \text{all diagonal matrices in } \mathbb{R}^{n \times n} \right\}
\]

\[
\text{Null}(\bar{D}) = \text{Range}(D)
\]

\[
\text{Range}(\bar{D}) = \text{Null}(D)
\]

\[(c) \quad D(P) = \lambda P \rightarrow \text{diag}(P) = \lambda P
\]

\[
\lambda = 1, \text{ } P \text{ is diagonal matrix}
\]

\[
\bar{D}(P) = \lambda P \rightarrow P - D(P) = \lambda P
\]

\[
\rightarrow D(P) = (1 - \lambda)P
\]

\[
\lambda = 0, \text{ } P \text{ is diagonal matrix}
\]

**Remark**: See Homework #2, Problem 3