Theory of Error Propagation

Consider a box having length L, width W and height H. How is the change of volume (V) of the box related to the changes in length, height and width? Since $V = LHW$, if only the length varies the change in volume is $\Delta V = \Delta LHW$, and $L\Delta HW$ if only the height changes and $LH\Delta W$ if only the width changes. If all three dimensions change, then the change in volume is simply the sum of these individual changes

$$\Delta V = \Delta LHW + L\Delta HW + LH\Delta W. \tag{1}$$

The variation of a function of several variables due to the change in only one of the variables is how we define partial derivatives. The first term in the equation above is simply the partial derivative of V with respect to L times the change in length ΔL. We can write the total differential change in volume in terms of partial derivatives as

$$dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial H} dH + \frac{\partial V}{\partial W} dW \,. \tag{2}$$

Now suppose $g(x,y,z)$ is a quantity that is a function of the measured variables x,y,z. Then g itself is uncertain due to the uncertainties of each of these measured values. A measure of the scatter of the individual measured values of each variable about their mean is provided by the average square deviation of g and is given by the sum of the square deviations divided by N. In fact, this becomes the commonly accepted definition of the square of the standard deviation, σ, when N is replaced by $N\text{-}1$.

$$\sigma_g{}^2 = \frac{1}{(N-1)} \sum_i \left(g_i - \overline{g}\right)^2 \text{(where } g_i \text{ is an individual value and } \overline{g} \text{ is the average value)} \tag{3}$$

This definition keeps σ meaningful for small populations (when $N = 1$, or a small number). Statistically, 68% of the measurements will fall within 1σ and 95% within 2σ. The variation (deviation from the mean) of the i^{th} value of g is related to the variations of the measured variables by

$$(g_i - \overline{g}) = (x_i - \overline{x})\frac{\partial g}{\partial x} + (y_i - \overline{y})\frac{\partial g}{\partial y} + (z_i - \overline{z})\frac{\partial g}{\partial z}. \tag{4}$$

Therefore, the square of the standard deviation is given by

$$\sigma_g^2 \;=\; \frac{1}{(N-1)} \sum_i \left[\left(x_i - \overline{x}\right)\frac{\partial g}{\partial x} + \left(y_i - \overline{y}\right)\frac{\partial g}{\partial y} + \left(z_i - \overline{z}\right)\frac{\partial g}{\partial x} \right]^2 \tag{5}$$

or after expansion by

$$\sigma_g^2 = \frac{1}{(N-1)}\left[\sum_i (x_i-\bar{x})^2\left(\frac{\partial g}{\partial x}\right)^2 + \sum_i (y_i-\bar{y})^2\left(\frac{\partial g}{\partial y}\right)^2 + \sum_i (z_i-\bar{z})^2\left(\frac{\partial g}{\partial z}\right)^2\right]. \qquad [6]$$
$$+ \frac{1}{(N-1)}\sum_i\left[(x_i-\bar{x})(y_i-\bar{y})\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + \textit{ other cross terms}\right]$$

The cross terms (second line in Eq. [6]) vanish in the summation if the variables are *linearly independent*. That is, if the variation in *x*, for instance, is independent of the variations in *y* and *z*. Note also that

$$\sigma_x^2 = \frac{1}{(N-1)}\sum_i (x_i-\bar{x})^2, \quad \sigma_y^2 = \frac{1}{(N-1)}\sum_i (y_i-\bar{y})^2, \text{ and } \quad \sigma_z^2 = \frac{1}{(N-1)}\sum_i (z_i-\bar{z})^2. \qquad [7]$$

Substituting the expressions from Eq. [7] into Eq. [6] yields

$$\sigma_g^2 = \sigma_x^2\left(\frac{\partial g}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial g}{\partial y}\right)^2 + \sigma_z^2\left(\frac{\partial g}{\partial z}\right)^2, \qquad [8]$$

which relates the standard deviation of the computed function σ_g to the standard deviations of the measured quantities. Since the standard deviation σ is related to the *standard error* α by the relationship $\alpha = \sigma/\sqrt{N}$, we can find the expected uncertainty of the computed quantity *g* from the uncertainties of the measured quantities *x,y,z*. This quantity is also known as the *standard deviation of the mean* or the *standard error of the mean*. Now lets see how this works for the different functional forms.

I. Addition and Subtraction

First we consider a quantity $g(x,y,z)$ that is a function of three independent parameters consisting of sums and differences

$$g(x,y,z) = ax + by + cz \quad \text{or} \quad g(x,y,z) = ax - by - cz. \qquad [9]$$

The partial derivatives are

$$\frac{\partial g}{\partial x} = a, \ \frac{\partial g}{\partial y} = \pm b, \text{ and } \frac{\partial g}{\partial z} = \pm c. \qquad [10]$$

The sign depends upon whether the terms are added or subtracted. We can use the expression derived above (Eq. 8) to find σ_g as a function of σ_x, σ_y and σ_z

$$\sigma_g{}^2 = \sigma_x{}^2\left(\frac{\partial g}{\partial x}\right)^2 + \sigma_y{}^2\left(\frac{\partial g}{\partial y}\right)^2 + \sigma_z{}^2\left(\frac{\partial g}{\partial z}\right)^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2. \qquad [11]$$

Note here that the minus signs vanish because of the squares. The square of the standard error is

$$\alpha_g^2 = \frac{\sigma_g^2}{N} = \frac{\left(a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2\right)}{N} = a^2\alpha_x^2 + b^2\alpha_y^2 + c^2\alpha_z^2. \qquad [12]$$

Thus, the uncertainty for measurements that have a functional dependence that involves sums and differences is

$$\alpha_g = \sqrt{a^2\alpha_x{}^2 + b^2\alpha_y{}^2 + c^2\alpha_z{}^2} = \sqrt{\frac{1}{N}\left(a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2\right)}\,. \qquad [13]$$

Rule for Addition and Subtraction:

The overall uncertainty is equal to the square root of the sum of the squares of the uncertainties of each of the individual terms.

II. Multiplication and Division

Next we consider a quantity $g(x,y)$ that is a function of two independent parameters consisting of a single multiplication or division

$$g(x,y) = \pm axy \quad \text{or} \quad g(x,y) = \pm ax/y\,. \qquad [14]$$

For the case of multiplication we have

$$\frac{\partial g}{\partial x} = \pm ay \text{ and } \frac{\partial g}{\partial y} = \pm ax \qquad [15]$$

$$\sigma_g^2 = \sigma_x^2\left(\frac{\partial g}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial g}{\partial y}\right)^2 = a^2y^2\sigma_x^2 + a^2x^2\sigma_y^2 \qquad [16]$$

and for division

$$\frac{\partial g}{\partial x} = \pm\frac{a}{y} \text{ and } \frac{\partial g}{\partial y} = \mp\frac{ax}{y^2} \qquad [17]$$

$$\sigma_g^2 = \sigma_x^2\left(\frac{\partial g}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial g}{\partial y}\right)^2 = \left(\frac{a}{y}\right)^2\sigma_x^2 + \left(\frac{ax}{y^2}\right)^2\sigma_y^2 \qquad [18]$$

Dividing by g^2 (in each case) results in the following expression

$$\frac{\sigma_g^2}{g^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} \qquad [19]$$

for both multiplication and division. Now, recalling that the standard error is $\alpha = \sigma/\sqrt{N}$, and that α_g/g is the fractional error of *g*, we have

$$\frac{\alpha_g}{g} = \sqrt{\left(\frac{\alpha_x}{x}\right)^2 + \left(\frac{\alpha_y}{y}\right)^2} \qquad [20]$$

for *both* multiplication and division.

Rule for Multiplication and Division

The Fractional Error of the quantity (fractional overall uncertainty) is equal to the square root of the sum of the squares of the individual fractional errors (note that α_x/x is the fractional error of *x*, etc.).

III. Powers

At first glance one may think that powers are just products and we proceed as described above for multiplication. For instance, the function $g(x,y) = Cxy^2 = Cxyy$, is a constant C times the product of three variables x, y and y, but the last two are obviously *not independent variables*. Therefore, the treatment above is no longer valid and we must develop the proper expression for variables raised to some power.

Consider a function of a single variable given by

$$g(x) = ax^{\pm b}, \qquad [21]$$

which has the following partial derivative

$$\frac{\partial g}{\partial x} = \pm abx^{\pm(b-1)}. \qquad [22]$$

Since

$$\sigma_g^2 = \sigma_x^2\left(\frac{\partial g}{\partial x}\right)^2 \text{ or } \sigma_g = \sigma_x \frac{\partial g}{\partial x} = \sigma_x abx^{\pm(b-1)}, \qquad [23]$$

we can obtain the fractional uncertainty by dividing Eq. [23] by $g = ax^{\pm b}$

$$\frac{\sigma_g}{g} = b\frac{\sigma_x}{x}. \qquad [24]$$

Recalling that the standard error is defined as

$$\alpha = \frac{\sigma}{\sqrt{N}} \qquad [25]$$

we obtain the fractional uncertainty in terms of the standard error

$$\frac{\alpha_g}{g} = b\frac{\alpha_x}{x}. \qquad [26]$$

<u>Rule for Powers</u>

For measurements that have the functional form $g(x,y,z) = Cx^m y^n z^p$, the fractional error on g is given by

$$\frac{\alpha_g}{g} = \sqrt{\left(m\frac{\alpha_x}{x}\right)^2 + \left(n\frac{\alpha_y}{y}\right)^2 + \left(p\frac{\alpha_z}{z}\right)^2}\,. \qquad [27]$$

Data Rejection

Sometimes when we make a series of measurements of a specific quantity one of the measured values disagrees strikingly with all the other measured values. When this happens the experimenter is presented with the situation where he/she must decide whether the anomalous measurement resulted from some mistake (glitch in the measurement system) and should be rejected or was a bona fide measurement that should be kept. If careful records were kept sometimes we can establish a definite cause for the anomalous measurement and therefore justifiably reject the measurement.

If an external cause can not be found for the anomalous result, then the truly honest course of action is to repeat the measurement many times. If the anomaly shows up again then hopefully the cause may be found. Either as a glitch in the measurement system or as a real physical effect. If the anomaly does not recur, then due to the increased number of measurements made there will be no significant difference in our final answer whether we include the anomaly or not.

If it is impossible to retake the measurements then the experimenter must decide whether or not to reject the anomaly by examining the measured data and the properties of a Gaussian distribution. The rejection of data is a subjective controversial question, on which experts disagree. The experimenter who rejects data may reasonably be accused of *fixing* his/her data. The situation is made worse by the possibility that the anomalous result may reflect some important physical effect. One criterion for rejecting suspect data is Chauvenet's criterion.

Chauvenet's Criterion for Data Rejection

Suppose we make N measurements $x_1, x_2, \ldots, x_N$ of the same quantity x.

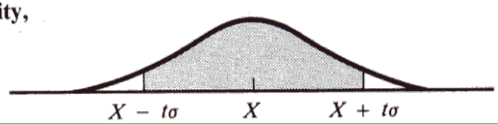
1. Using ***all*** the values of the N measurements made calculate the mean $(\bar{x})$ and standard deviation (σ_x). If one of the measurements (call it $x_{suspect}$) greatly differs from $\bar{x}$ and looks suspicious, then calculate
$$t_{suspect} = \frac{x_{suspect} - \bar{x}}{\sigma_x},$$
the number of standard deviations by which $x_{suspect}$ differs from $\bar{x}$.

2. We next find the probability $P(\text{outside } t_{suspect}\sigma_x)$ that a legitimate measurement will differ from $\bar{x}$ by $t_{suspect}$ or more standard deviations.
$$P(\text{outside } t_{suspect}\,\sigma_x) = 1 - P(\text{within } t_{suspect}\,\sigma_x)$$

3. Finally, we multiply by N, the total number of measurements, to arrive at
$$n(\text{worse than } x_{suspect}) = N\,P(\text{outside } t_{suspect}\,\sigma_x)$$
This n is the number of measurements expected to be at least as bad as $x_{suspect}$.

If n is less than $\frac{1}{2}$, then $x_{suspect}$ fails Chauvenet's criterion and is rejected.

Table A. The percentage probability,
$P(\text{within } t\sigma) = \int_{X - t\sigma}^{X + t\sigma} f_{X,\sigma}(x)\,dx$,
as a function of t.



t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.1	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.2	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.3	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.4	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.5	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.6	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.7	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.8	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.9	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.0	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.1	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.2	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.3	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.4	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.5	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.6	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.7	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.8	92.81	92.97	93.12	93.28	93.42	93.57	93.71	93.85	93.99	94.12
1.9	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.0	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.1	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.2	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80
2.3	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.4	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72
2.5	98.76	98.79	98.83	98.86	98.89	98.92	98.95	98.98	99.01	99.04
2.6	99.07	99.09	99.12	99.15	99.17	99.20	99.22	99.24	99.26	99.29
2.7	99.31	99.33	99.35	99.37	99.39	99.40	99.42	99.44	99.46	99.47
2.8	99.49	99.50	99.52	99.53	99.55	99.56	99.58	99.59	99.60	99.61
2.9	99.63	99.64	99.65	99.66	99.67	99.68	99.69	99.70	99.71	99.72
3.0	99.73									
3.5	99.95									
4.0	99.994									
4.5	99.9993									
5.0	99.99994									